Resugence
and quantum topology

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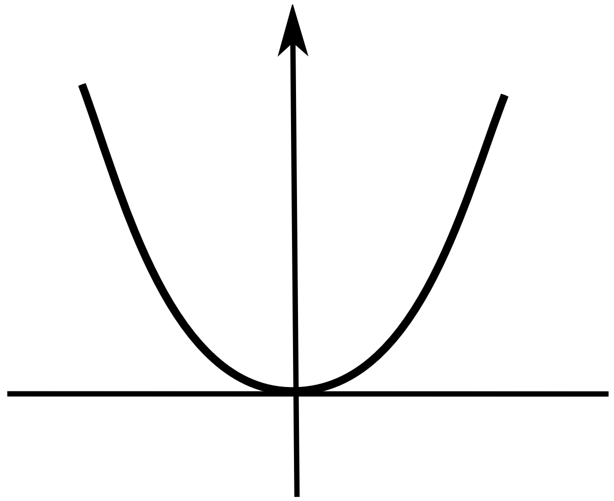
Perturbation theory and its discontents

Perturbation theory in a small parameter remains one of the most fruitful approaches in physics, in the absence of exact solutions.

A simple example is the ground state energy of the quartic oscillator in quantum mechanics



$$\mathsf{H} = \frac{\mathsf{p}^2}{2} + \frac{\mathsf{x}^2}{2} + g\mathsf{x}^4$$

$$E_0(g) \sim \frac{1}{2} + \frac{3}{4}g - \frac{21}{8}g^2 + \cdots$$

However, perturbative series must be handled with care. Typically, the coefficients grow **factorially**

$$E_0(g) \sim \sum_n a_n g^n \qquad a_n \sim n!$$

Therefore, these series have zero radius of convergence and do not lead (at least immediately) to functions.

Physicists and mathematicians have developed various tricks to make sense of these divergent series and extract information from them. The **theory of resurgence** emerged in the 1970s-1980s as a general framework to address these issues.

Hidden structures in perturbation theory

It turns out that factorially divergent series encode a wealth of information about the full theory. In particular, they secretly know about non-perturbative sectors. I will call this hidden information the **resurgent structure** of the series.

As I will try to show in this talk, the resurgent structure of perturbative series involves in some cases a hidden **integrality structure** which can be physically interpreted in terms of counting BPS states in a “dual” theory.

Two very interesting examples of such a situation are complex Chern-Simons theory and topological string theory. I will focus my talk on the first example.

This is based on work with Stavros Garoufalidis, Jie Gu and Campbell Wheeler

Decoding perturbation theory

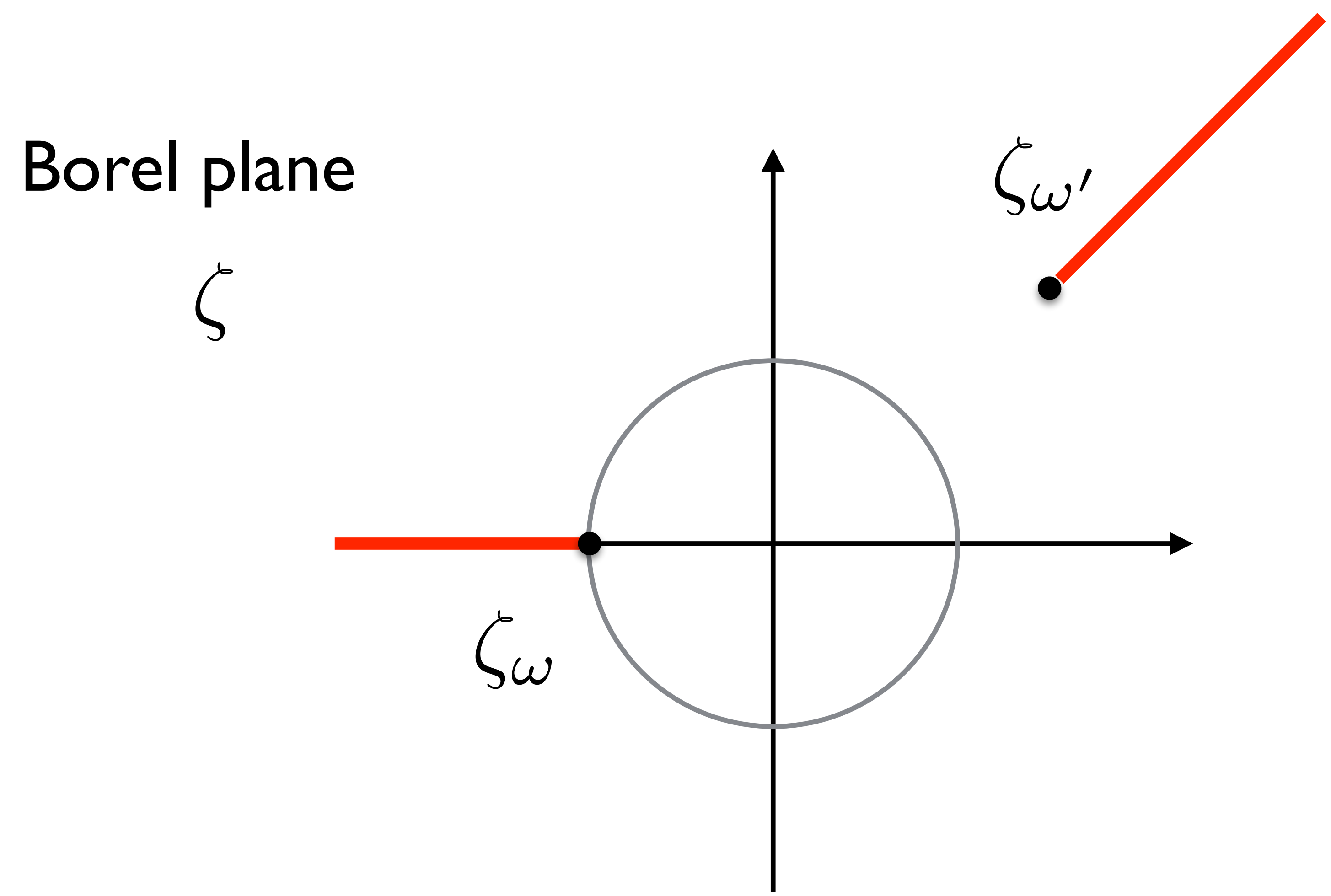
Let us consider a formal power series with factorially growing coefficients

$$\varphi(z) = \sum_{n\geq 0} a_n z^n \qquad\qquad a_n \sim n!$$

These are sometimes called Gevrey-1 series. The first step in resurgence is the **Borel transform**, a deceptively simple way of transforming these series into "nice" functions

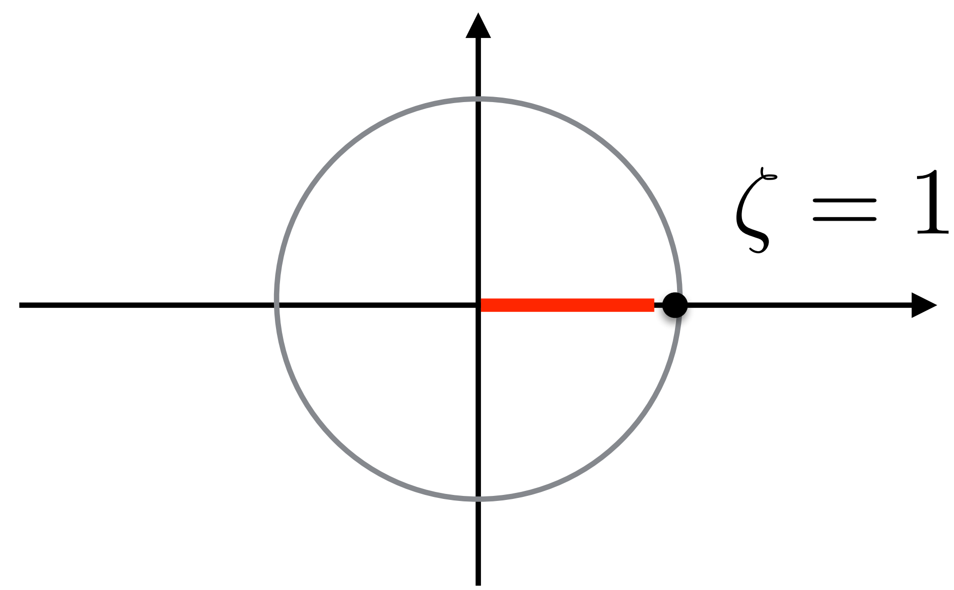
$$\varphi(z) = \sum_{n\geq 0} a_n z^n \quad \xrightarrow{\text{Borel transform}} \quad \widehat{\varphi}(\zeta) = \sum_{n\geq 0} \frac{a_n}{n!} \zeta^n$$

The Borel transform $\widehat{\varphi}(\zeta)$ is analytic at the origin. Very often it can be analytically continued to the complex plane, displaying a set of **singularities** (poles, branch cuts)



Example: $$\varphi(z) = \sum_{k\geq 0} k! z^k$$

$$\widehat{\varphi}(\zeta) = \sum_{k\geq 0} \zeta^k = \frac{1}{1-\zeta}$$



To extract the hidden information, we have to consider the expansion of the Borel transform around each of its singularities. These leads to **new formal power series**.

Let us consider for simplicity the so-called **simple resurgent functions**, where singularities are logarithmic branch cuts. The expansion around a singularity at $\zeta = \zeta_\omega$ has the form

$$\widehat{\varphi}(\zeta) = -\mathsf{S}_\omega \, \widehat{\varphi}_\omega(\zeta - \zeta_\omega) \frac{\log(\zeta - \zeta_\omega)}{2\pi \mathrm{i}} + \text{regular}$$

The function $\widehat{\varphi}_\omega(\xi)$ is typically analytic at the origin, but we can think about it as the Borel transform of a **new power series** associated to the singularity:

$$\varphi_\omega(z) = \sum_{n \geq 0} a_{n,\omega} z^n$$

The constant S_ω is called a **Stokes constant** and plays an important role in the theory. Its value depends on the normalization of $\varphi_\omega(z)$

We can repeat the same analysis for the power series found in this way, and generate further series. At the end, we obtain **a set of formal power series** associated to the original power series, which we will call sometimes the "minimal resurgent structure" for $\varphi(z)$

$$\varphi(z) \longrightarrow \mathfrak{B}_\varphi = \{\varphi_\omega(z)\}_{\omega\in\Omega}$$

We also have a **matrix of Stokes constants** defined by

$$\widehat{\varphi}_\omega(\zeta_{\omega'} + \xi) = -\mathsf{S}_{\omega\omega'} \frac{\log(\xi)}{2\pi \mathrm{i}} \widehat{\varphi}_{\omega'}(\xi) + \mathrm{regular}$$

It turns out that, in some important cases, Stokes constants are **integers**. The appearance of integer Stokes constants means that we have a correspondence between

analytic structure ⟷ integrality structure

This is a generalization of a similar relationship between Riemann-Hilbert problems and integrality structures developed by e.g. [Gaiotto-Moore-Neitzke, Bridgeland]

An elementary example: the Airy functions

The formal power series underlying the Airy "Ai" function is

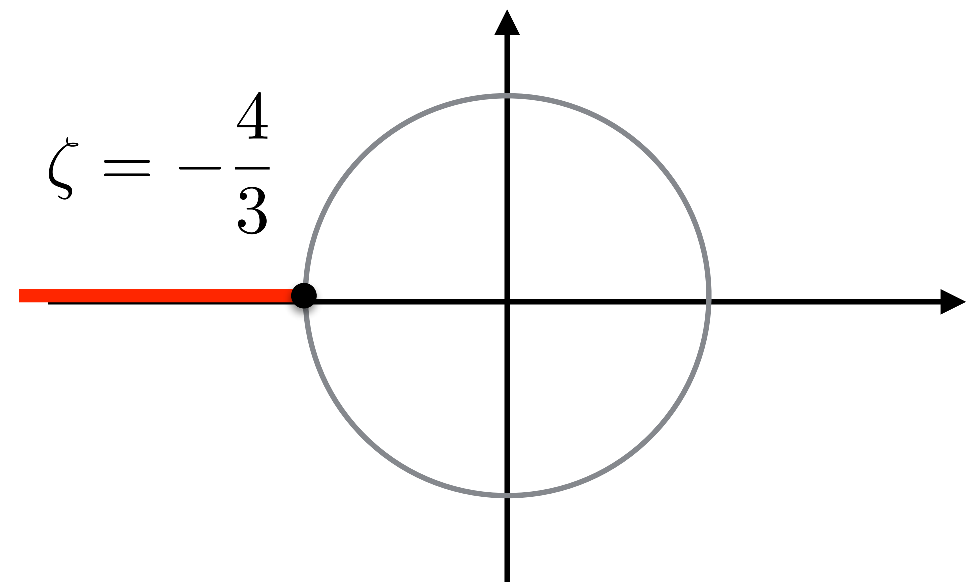
$$\varphi_1(z) = \sum_{n \geq 0} \frac{1}{2\pi} \left(-\frac{3}{4}\right)^n \frac{\Gamma(n+\frac{5}{6})\Gamma(n+\frac{1}{6})}{n!} z^n$$

In this case the resurgent structure can be worked out in detail, since the Borel transform is given explicitly by

$$\widehat{\varphi}_1(\zeta) = {}_2F_1\left(\frac{1}{6}, \frac{5}{6}, 1; -\frac{3\zeta}{4}\right)$$

This a simple resurgent function, with a log singularity along the negative real axis:

Borel plane:



By studying the expansion around the singularity, one finds the other formal power series involved in the game, underlying the "Bi" Airy function

$$\varphi_2(z) = \sum_{n\geq 0} \frac{1}{2\pi}\left(\frac{3}{4}\right)^n \frac{\Gamma(n+\frac{5}{6})\Gamma(n+\frac{1}{6})}{n!} z^n$$

The Stokes constants are $\mathsf{S}_{12} = \mathsf{S}_{21} = \mathrm{i}$

Resurgence and saddle points

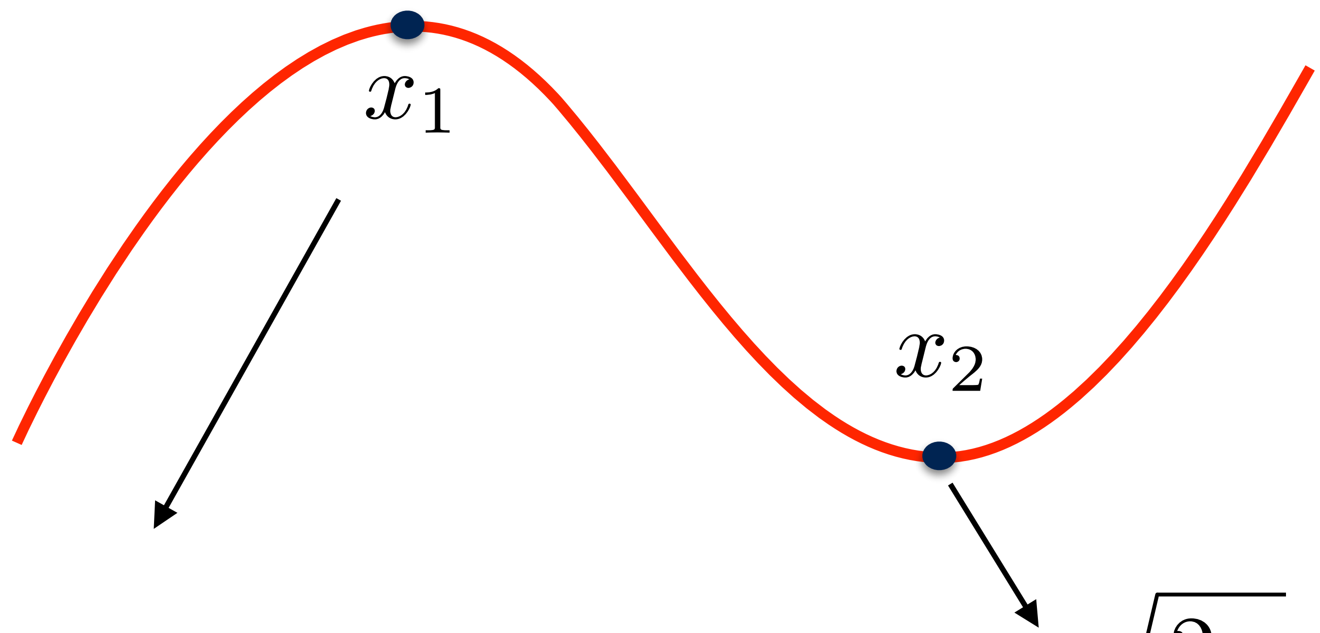
The Airy series are a particular case of a more general class of examples: the formal power series appearing in the study of the saddle-point approximation to integrals of the form

$$\mathcal{I}(\hbar) = \int \mathrm{d}x \, \mathrm{e}^{-S(x)/\hbar}$$

Saddle-points satisfy the "equation of motion"

$$S'(x) = 0$$

By doing a Gaussian expansion around these points, we typically obtain Gevrey-1 series

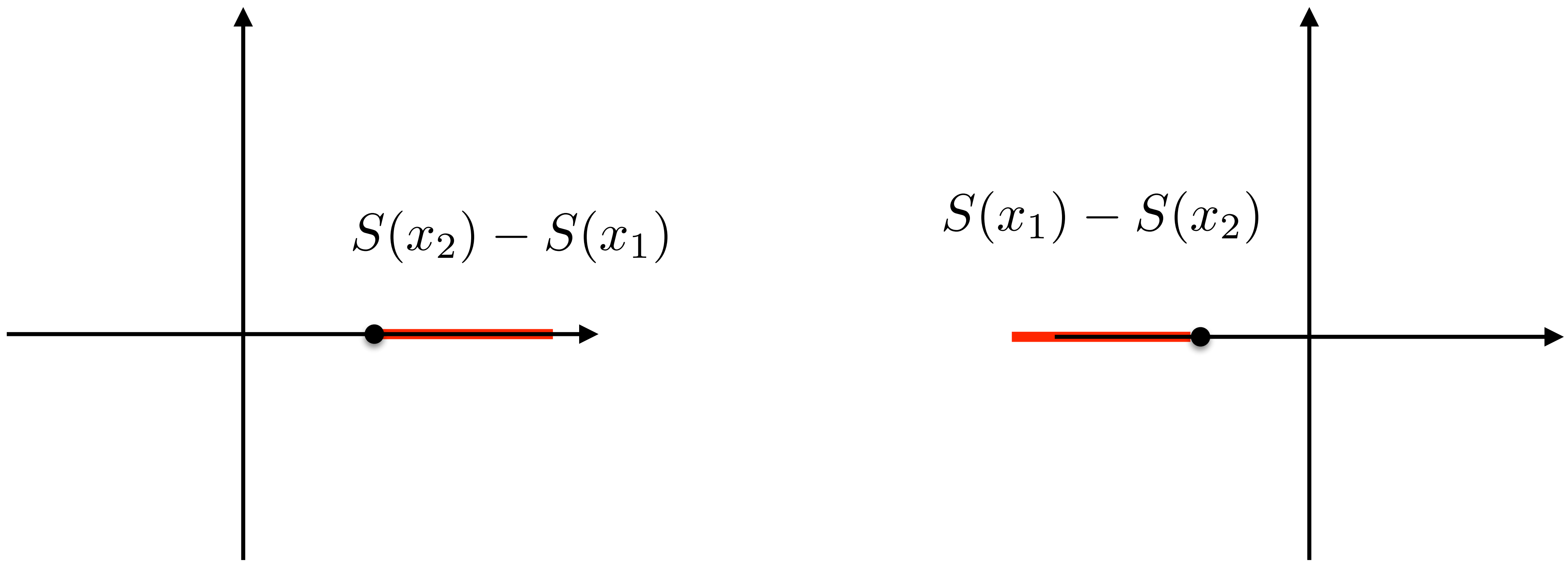


$$\mathrm{i}\sqrt{\frac{2\pi}{\hbar}}\,\mathrm{e}^{-S(x_1)/\hbar}\varphi_1(\hbar) \qquad \sqrt{\frac{2\pi}{\hbar}}\,\mathrm{e}^{-S(x_2)/\hbar}\varphi_2(\hbar)$$

It turns out that the singularities of the Borel transform of $\varphi_i(\hbar)$ are located at

$$S(x_j) - S(x_i), \qquad j \neq i$$

The Borel transform of the series at one saddle-point sees the contribution of other saddle-points!



$\widehat{\varphi}_1(\zeta)$ $\widehat{\varphi}_2(\zeta)$

This result generalizes to path integrals in quantum mechanics and in some quantum field theories

Complex Chern-Simons theory

A rich source of perturbative series (and beautiful mathematics!) is (complex) Chern-Simons (CS) theory on a 3-manifold *M*.

CS theory is a gauge theory for a G-connection *A* on a three manifold *M*. Its action has the form:

$$S(A) = \frac{1}{4\pi}\int_M \mathrm{Tr}\left(A\wedge \mathrm{d}A + \frac{2}{3}A\wedge A\wedge A\right)$$

The partition function of this theory is formally defined by the path integral

$$Z_M(\tau) = \int \mathcal{D}A\, \mathrm{e}^{\mathrm{i}S(A)/\tau}$$

The Chern-Simons action is multivalued, and is only defined up to an integer multiple of 2π. This will play a role later.

The solutions of the classical EOM are just flat connections

$$F(A) = \mathrm{d}A + A \wedge A = 0$$

We will focus on the case in which *G*= *SL(2,C)*, a complex gauge group. This theory has been much studied in the last years [Witten, Kashaev, Garoufalidis, Gukov, Hikami, Dimofte, Andersen…]

In this case, τ is naturally complex

When M is the complement of a hyperbolic knot K inside the three-sphere, it has been argued that the partition function of the theory can be reduced to a finite-dimensional **state-integral**

$$Z_K(\tau) = \int \mathrm{e}^{-W(\mathbf{x};\tau)/\tau} \mathrm{d}\,\mathbf{x}$$

The integrand involves in a crucial way Faddeev's quantum dilogarithm

Saddle-points of the state-integral correspond to flat complex connections σ on *M*. The expansions around these connections have the form

$$\exp\left(\frac{V(\sigma)+\mathrm{i}\mathcal{C}(\sigma)}{2\pi\tau}\right)\varphi_\sigma(\tau)$$

$$\varphi_\sigma(\tau)=\sum_{n\geq 0} a_n^\sigma \tau^n$$

factorially divergent series!

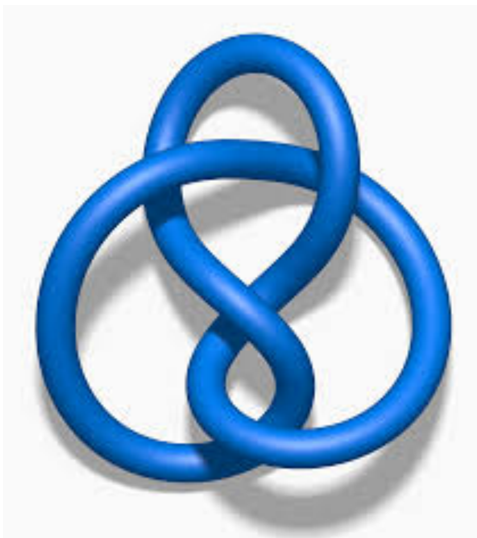
$$a_n^\sigma \sim n!$$

Among these connections there is always the "geometric connection" *g* (corresponding to the geodesically complete hyperbolic metric on *M*), and its conjugate *c*, with

$$V(g,c)=\pm V$$

hyperbolic volume

These saddle points lead to "classical" singularities in the Borel plane.



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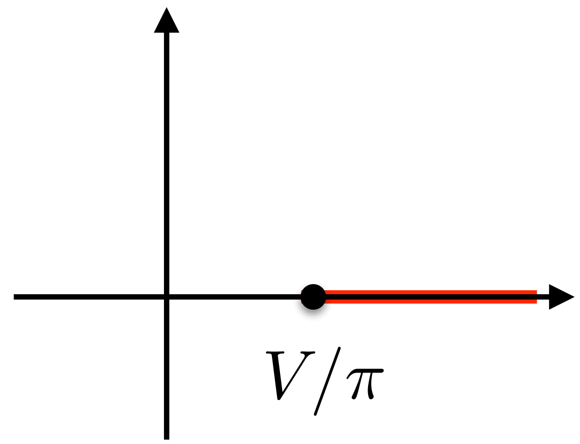
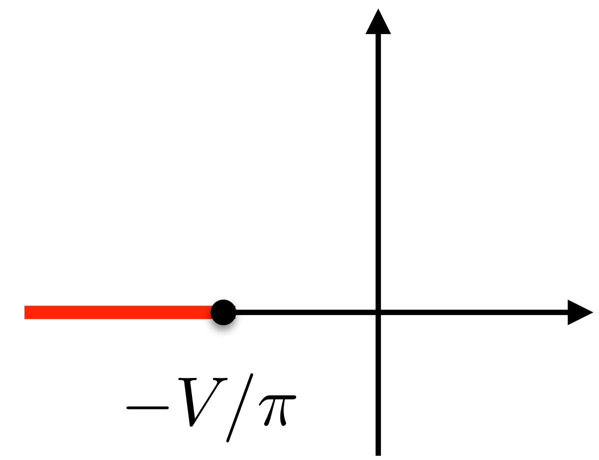


$\varphi_g(\tau)$



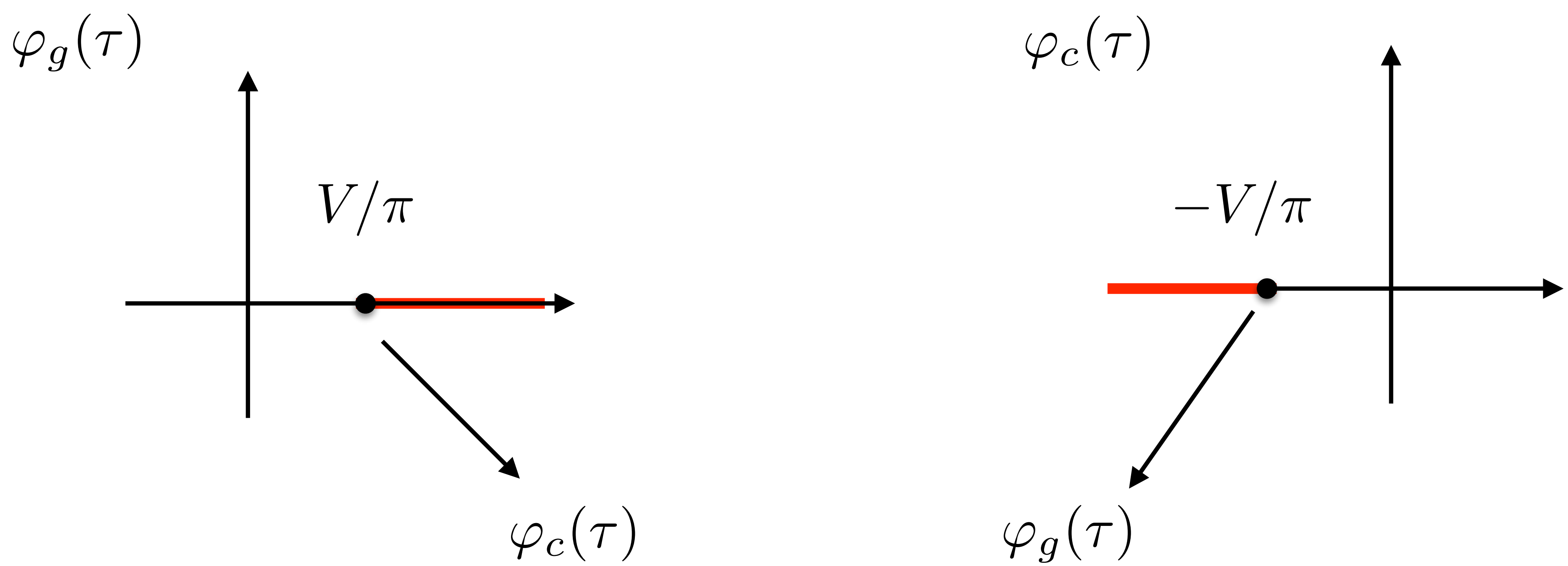
$\varphi_c(\tau)$

$$\varphi_g\left(\frac{\tau}{2\pi \mathrm{i}}\right) = \frac{1}{3^{1/4}}\left(1 + \frac{11\tau}{72\sqrt{-3}} + \frac{697\tau^2}{2\,(72\sqrt{-3})^2} + \frac{724351\tau^3}{30\,(72\sqrt{-3})^3} + \cdots\right)$$

=all-orders
Kashaev invariant

$$\varphi_c(\tau) = \varphi_g(-\tau)$$

As we explained before, to each of these singularities we can associate a formal power series and a Stokes constant. In this case, each power series sees the other one, as in the Airy function example



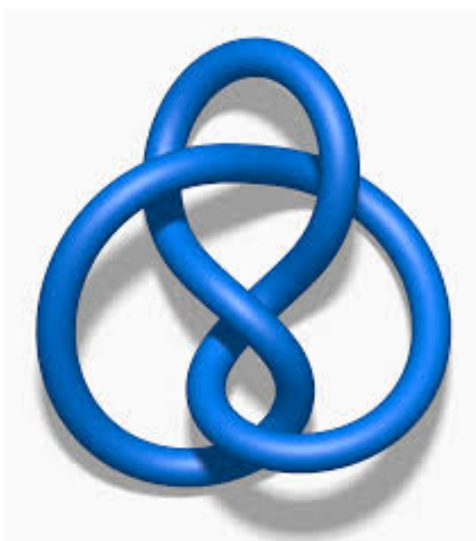
$$\mathsf{S}_{gc} = -\mathsf{S}_{cg} = 3$$

These "classical" singularities and their Stokes constants were analyzed in e.g. [Gukov-M.M.-Putrov, Gang-Hatsuda, Garoufalidis-Zagier]

However, due to multivaluedness of the CS action and of the state integral potential, there are **infinite towers of additional singularities** [Garoufalidis, Witten, Gukov-M.M.-Putrov,…], corresponding to the "actions"

$$\frac{V(\sigma) + \mathrm{i}\mathcal{C}(\sigma)}{2\pi} + 2\pi \mathrm{i} n \qquad n \in \mathbb{Z}$$

The actual picture is rather



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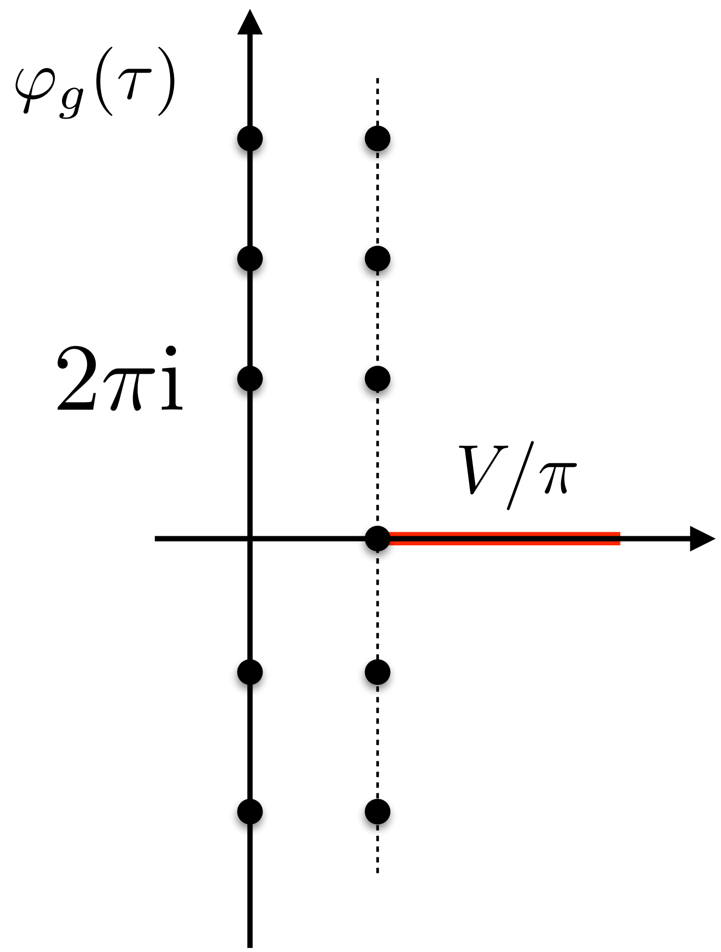
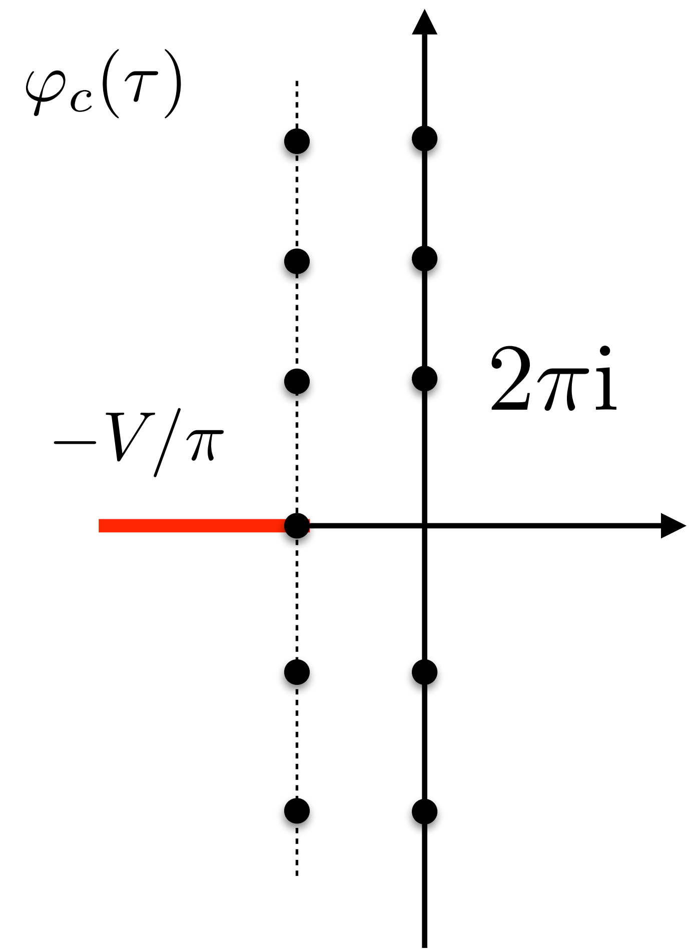
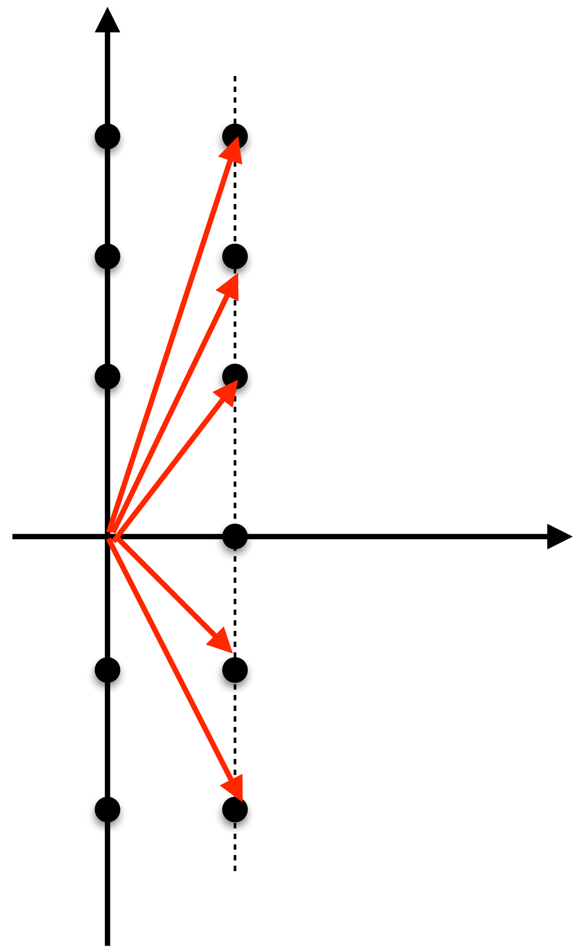




We call these infinite towers "peacock patterns."

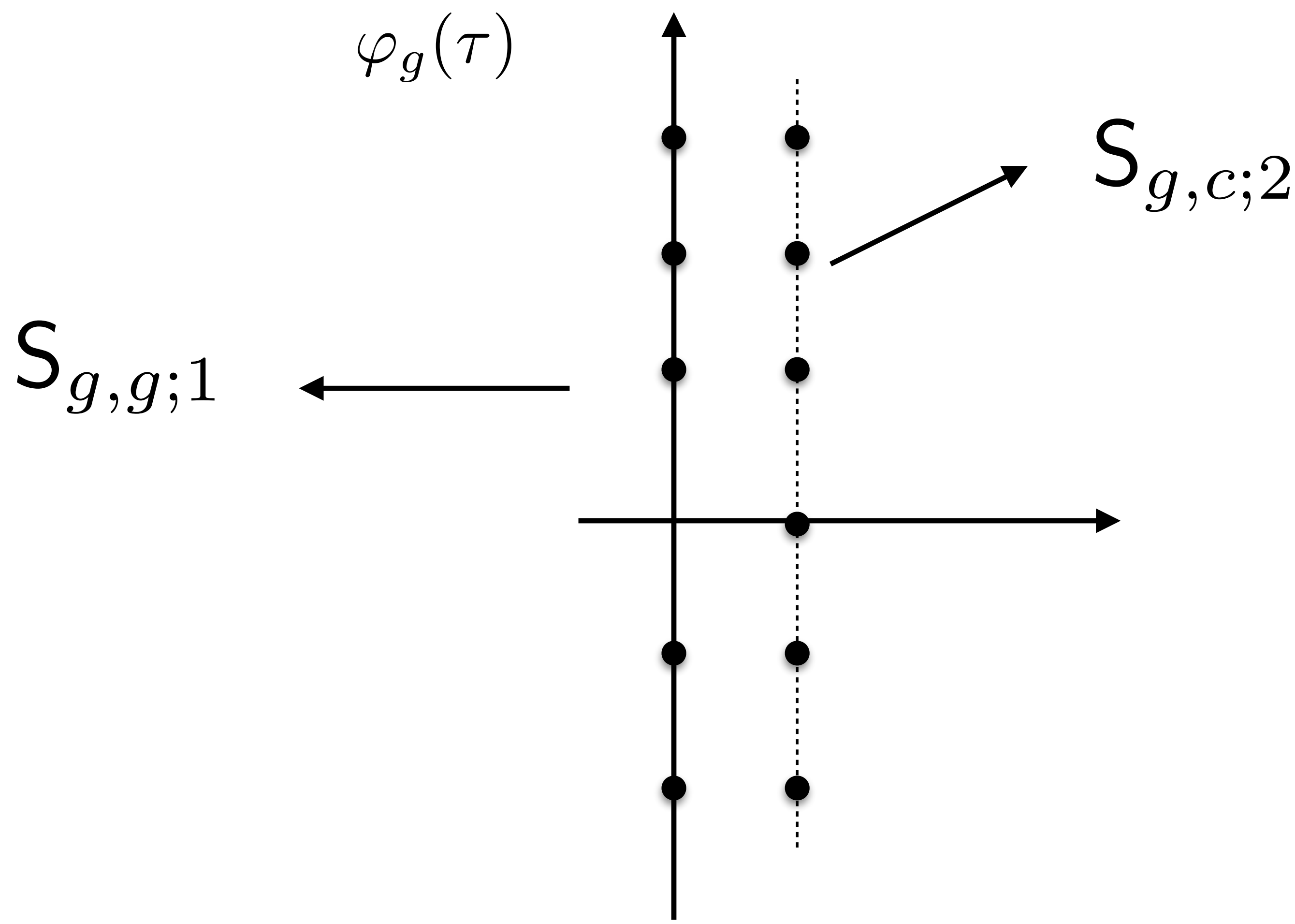


Similar infinite towers of singularities appear in other contexts, like topological string theory [Pasquetti-Schiappa, Couso-M.M.-Schiappa]

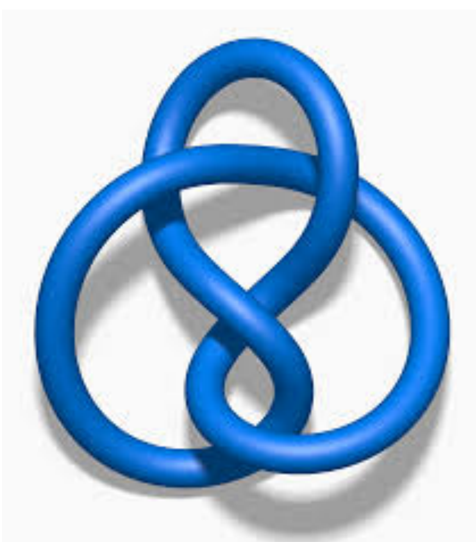
One could think that these towers are "trivial", and indeed the power series associated to the singularities are always

$$\varphi_{c,g}(\tau)$$

However, the Stokes constants turn out to be different! We will label them as $\mathsf{S}_{\sigma,\sigma';n}$



Explicit computations show that these Stokes constants provide **a highly non-trivial collection of integer invariants of the knot!**



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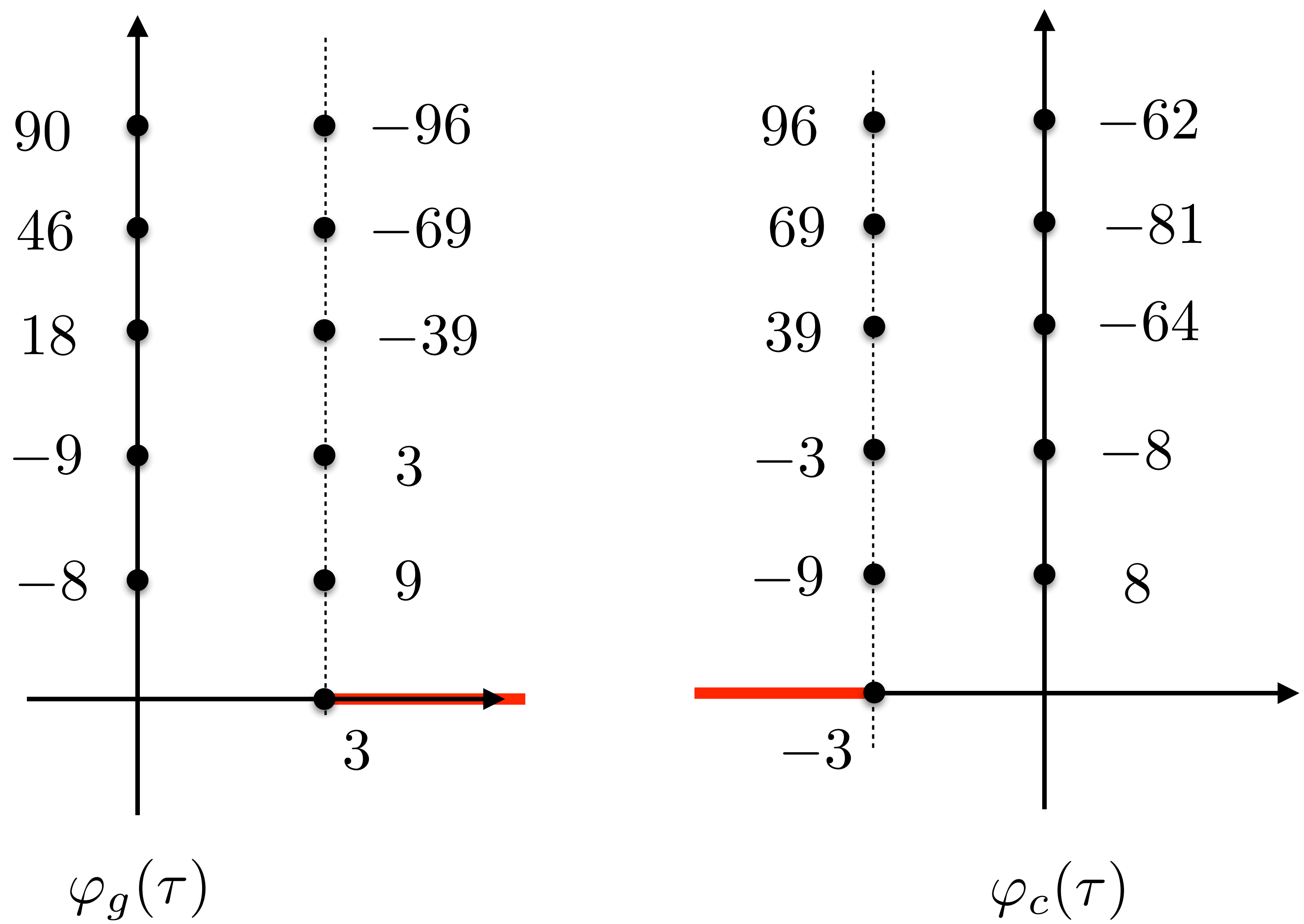
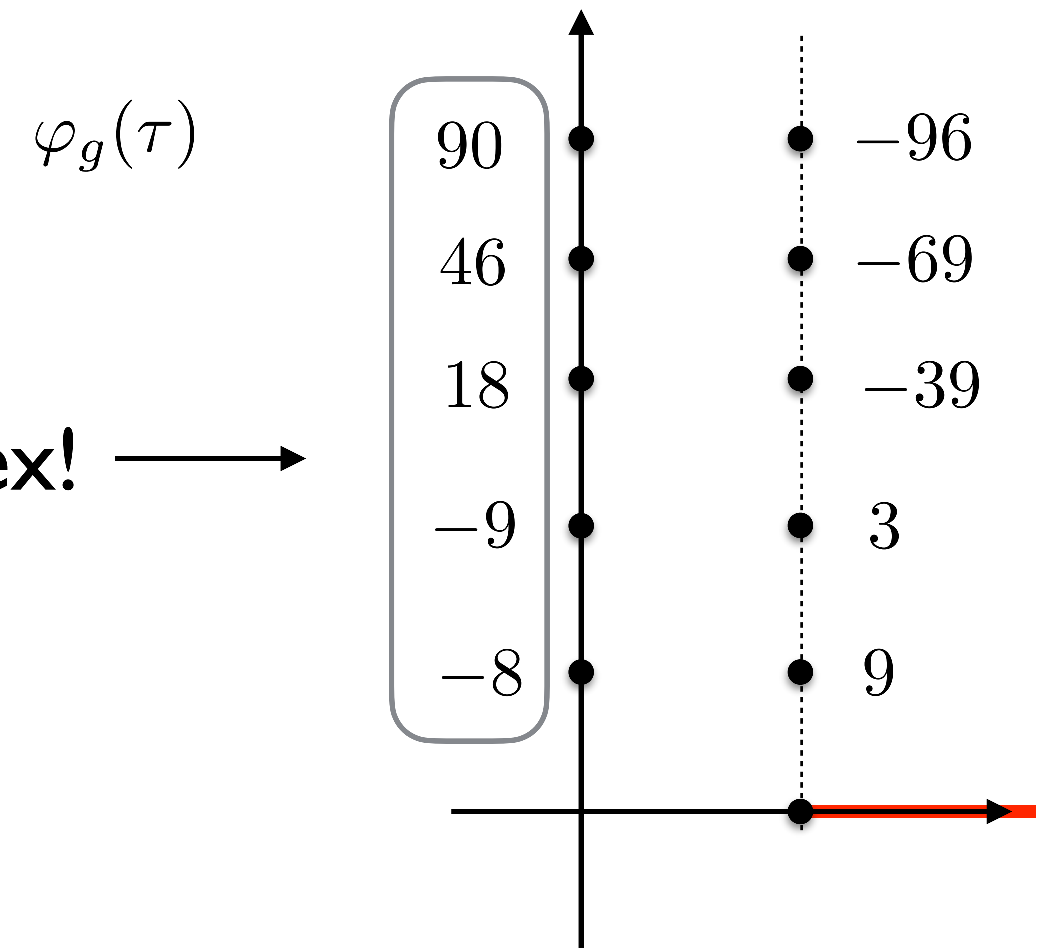


Can we find a systematic description of these integers?

DGG index! →



$$\mathrm{Tr}_{\mathcal{H}_{m=0}}(-1)^F q^{R/2+j_3} = 1 - 8q - 9q^2 + 18q^3 + 46q^4 + \cdots$$

This index counts BPS states in a 3d SCFT "dual" to the hyperbolic knot [Dimofte-Gaiotto-Gukov]

In particular, we find **completely explicit (conjectural) expressions** for the "Stokes *q*-series" which collect all Stokes constants

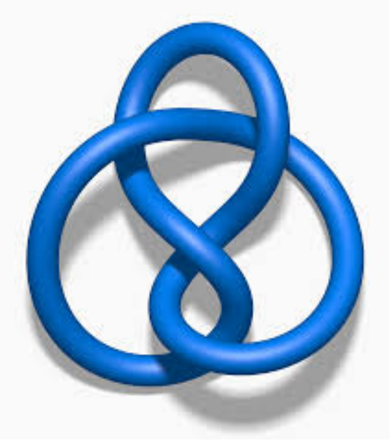
$$\mathcal{S}^{\pm}_{\sigma\sigma'}(q) = \sum_{n\geq 1} \mathsf{S}_{\sigma,\sigma';n} q^{\pm n}$$

These *q*-series are related to the "holomorphic block decomposition" of the state integral [Beem-Dimofte-Pasquetti, Garoufalidis-Kashaev, Garoufalidis-Zagier] and to more general versions of the DGG index (i.e. with magnetic charges), therefore to a counting of BPS states

This provides in particular a full "decoding" of the state integral invariant in terms of (Borel resummed) perturbative series, plus non-perturbative effects

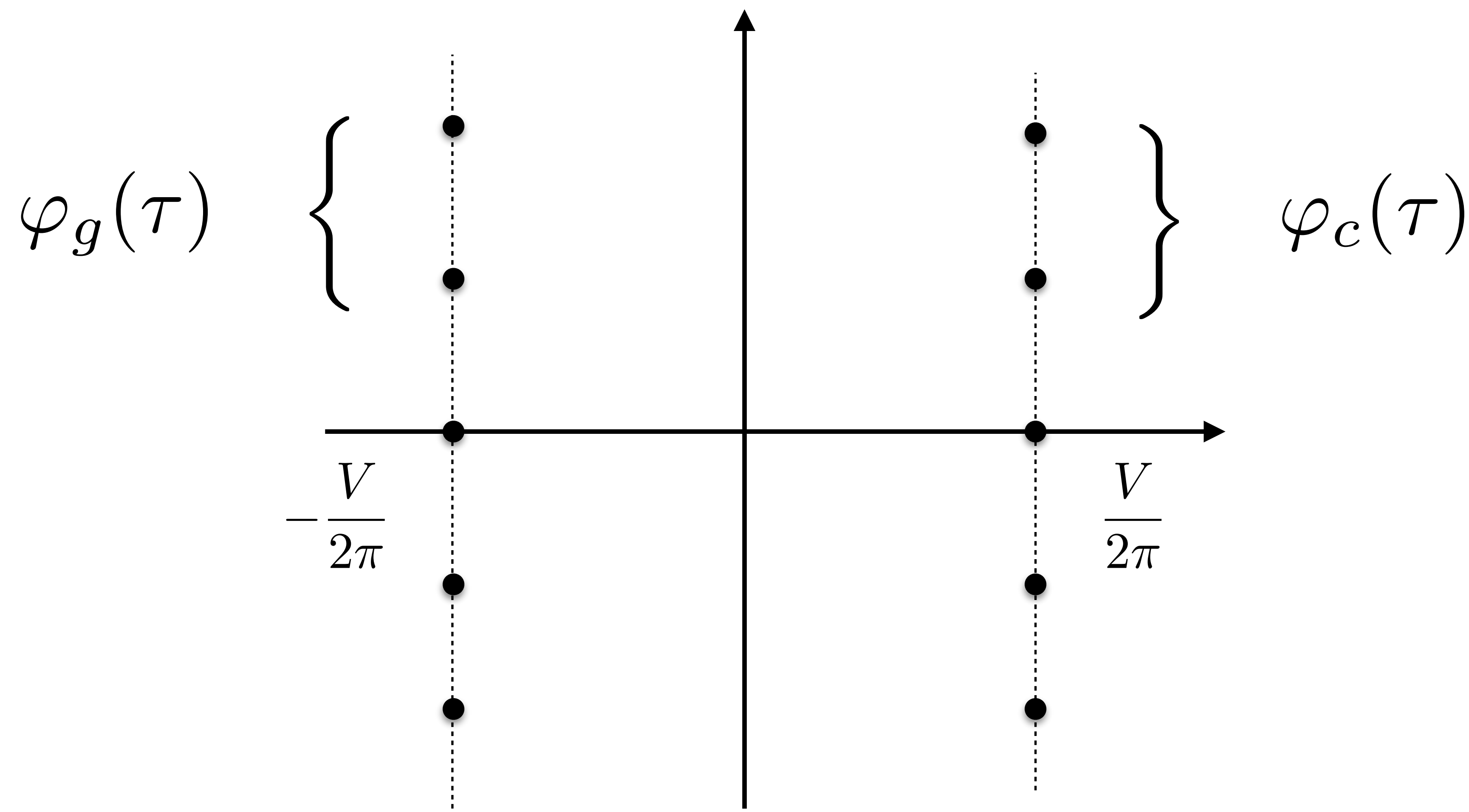
What about the trivial connection?

One well-known shortcoming of state-integrals is that they do not give information about the simplest flat connection, namely the trivial one *A=0*. There is however a well-known power series attached to it, which can be obtained from *SU(2)* Chern-Simons theory



$$\Phi_0(\tau) = 1 + 4\pi^2\tau^2 + \frac{188\pi^4\tau^4}{3} + \frac{98888\pi^6\tau^6}{45} + \dots$$

What is the resurgent structure associated to this series?



Note that the trivial connection "sees" the other ones, but not the other way around! [Garoufalidis-Zagier, Witten]

A complete theory must include the additional Stokes constants associated to the trivial connection, i.e. the Stokes matrix should be enlarged

We have been able to determine (conjecturally) the enlarged Stokes matrix for the very first hyperbolic knots:

$$\mathcal{S}^{+}(q) = \begin{pmatrix} 1 & q-6q^2-22q^3-13q^4+O\left(q^5\right) & q+11q^2+24q^3+5q^4+O\left(q^5\right) \\ 0 & 1-8q-9q^2+18q^3+46q^4+O\left(q^5\right) & 9q+3q^2-39q^3-69q^4+O\left(q^5\right) \\ 0 & -9q-3q^2+39q^3+69q^4+O\left(q^5\right) & 1+8q-8q^2-64q^3-81q^4+O\left(q^5\right) \end{pmatrix}$$

The two new entries in the top row are closely related to (and generalize) the Gukov-Manolescu q-series of knots.

Conclusions and open questions

Resurgence can be used to extract precious information from perturbative series. In some cases, it leads to non-trivial integer invariants. This is **a new route to integrality**, different from previous ones (like radial asymptotics of q-series).

Our results determine (at least conjecturally) the **complete** resurgent structure of complex Chern-Simons theory for hyperbolic knots, and indicate a close relationship between Stokes constants and BPS counting in the "dual" 3d theory.

Similar structures appear in topological string theory on CY manifolds, and lead to (conjecturally) new integer invariants

Thank you for your attention!

Resurgent structures in topological string theory

It turns out that similar structures appear in topological string (TS) theory on Calabi-Yau (CY) manifolds.

What is the right formal power series to start with? The all-genus expansion of the topological string has the form

$$F^{\mathrm{LR}}(t,\tau) \sim \sum_{g\geq 0} F_g^{\mathrm{LR}}(t)\tau^{2g-2}$$

τ =string coupling constant

We assume for simplicity that there is a single CY modulus

The genus *g* free energies encode Gromov-Witten invariants

$$F_g^{\mathrm{LR}}(t) = \sum_{d\geq 1} N_{g,d}\, \mathrm{e}^{-dt}$$

It is known that, at fixed *t* [Shenker], $F_g(t) \sim (2g)!$

so in principle we could use the theory of resurgence to analyze singularities, Stokes constants, etc. This is a difficult problem, since the coefficients are themselves functions of a parameter t, the CY modulus. The study of these type of series is called **parametric resurgence**.

$$t = N\tau$$

This is similar to a large *N* gauge theory, where one has series of the form

$$F_g(t) N^{2-2g}$$

The modulus of the CY can be interpreted as a 't Hooft parameter, as in the Gopakumar-Vafa duality

$$t = N\tau$$

Therefore, to obtain the analogue of a conventional perturbative series in a gauge theory, as we did in Chern-Simons, we should fix N and take the perturbative limit

$$\tau \to 0$$

This is the "tensionless" limit of the TS. However, in this limit t vanishes, and this is incompatible with the large radius regime!

We have to consider topological string theory in a different regime, near the **conifold** point. There is a local coordinate λ in the moduli space of the CY which vanishes at that point.

$$Z(\lambda, \tau) = \exp\left(\sum_{g\geq 0} F_g^{\rm c}(\lambda)\tau^{2g-2}\right)$$

The conifold free energies are related to the conventional large radius energies by a generalized electric-magnetic duality. This is similar to Seiberg-Witten theory

large radius $\longleftrightarrow$ electric frame

conifold $\longleftrightarrow$ magnetic frame

$$\lambda \leftrightarrow a_D$$

We will think about the conifold coordinate as a 't Hooft parameter

$$\lambda = N\tau \qquad\qquad N = 1, 2, \cdots$$

This defines an **infinite family of formal power series** (times an exponential)

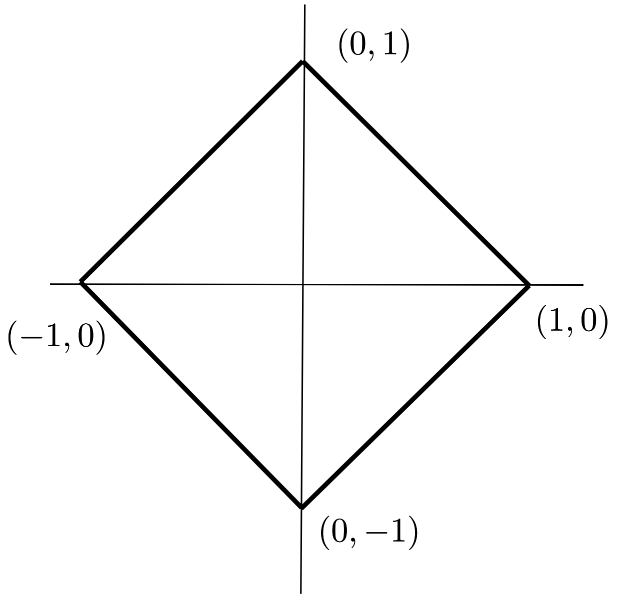
$$Z_N(\tau) = Z(\lambda = N\tau, \tau) \; = \mathrm{e}^{\mathcal{A}_N/\tau} \varphi_N(\tau)$$

Conjecture: these formal power series lead to resurgent structures with **integer** Stokes constants

A toric example

The calculation of the $Z_N(\tau)$ is not straightforward, since we need all-genus information For this reason, although we expect this conjecture to be true in general, we have evidence in the case of **toric** CYS.

Let us consider local $\mathbb{F}_0$, a favourite example for topological string theories

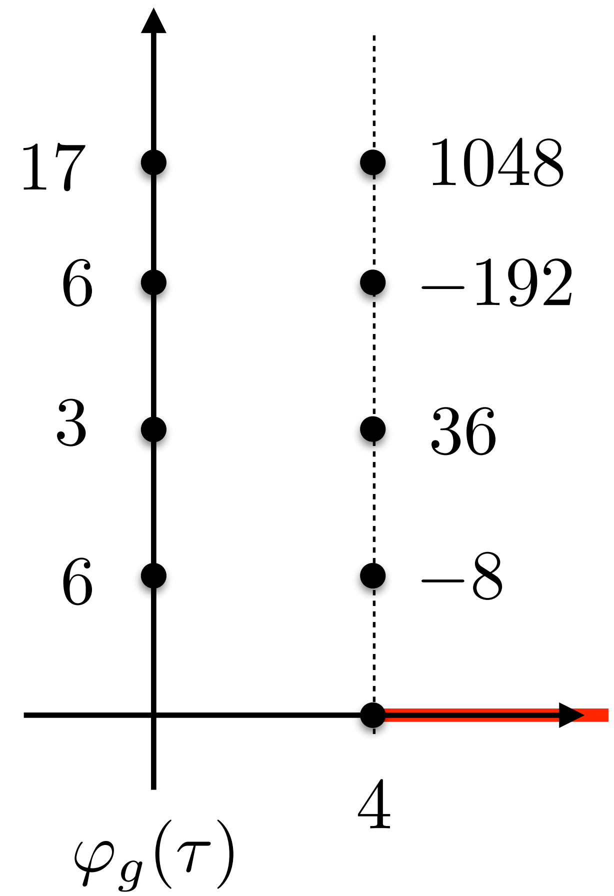


$$N = 1 \qquad Z_1(\tau) \sim \exp\left(\frac{2C}{\pi\tau}\right) \underbrace{\left(1 + \frac{\pi\tau}{24} + \cdots\right)}_{\varphi_g(\tau)}$$

It turns out that, as in the figure eight knot, there is another formal power series involved in the resurgent structure

$$\varphi_c(\tau) = \varphi_g(-\tau)$$

In this case, the additional towers of additional singularities are spaced by $\pi n i$



The precise meaning of these integers (for which we have explicit conjectures) is still not known, but it is natural to suspect that they are related to the counting of the BPS states in the CY.

Let us recall that the power series appearing in complex Chern-Simons theory have an uplifting to a non-perturbative object: the state-integral invariant.

In the case of $Z_N(\tau)$, there is also a conjectural non-perturbative uplifting: according to the TS/ST correspondence of [Grassi-Hatsuda-M.M.], they are asymptotic expansions of **spectral traces of quantum mechanical operators**. These operators are obtained by Weyl quantization of the mirror curve to X

$$\mathrm{Tr}_{\Lambda^N(L^2(\mathbb{R}))} \rho_X^{\otimes N} \sim Z_N(\tau)$$

local $\mathbb{F}_0$:

$$\rho_{\mathbb{F}_0}^{-1} = \mathrm{e}^{\mathsf{x}} + \mathrm{e}^{-\mathsf{x}} + \mathrm{e}^{\mathsf{y}} + \mathrm{e}^{-\mathsf{y}}$$

$$[\mathsf{x}, \mathsf{y}] = \mathrm{i}\hbar \qquad \tau = -\frac{1}{\hbar}$$

Many open questions:

Can we reformulate the state-integral invariants/spectral traces in terms of a Riemann-Hilbert problem?

Can we prove our conjectures or justify them physically [G. Moore and collaborators, work in progress]?

Can we develop a theory of parametric resurgence for topological strings, including the full dependence on the moduli [Couso-M.M.-Schiappa, Alim et al.]?

Can we develop a similar theory for topological strings on compact CYs?