An introduction to decomposition

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An overview of hep-th/0502027, 0502044, 0502053, 0606034, … (many …),
& recently arXiv: 2101.11619, 2106.00693, 2107.12386, 2107.13552, 2108.13423,
2204.09117, 2204.13708 & to appear   w/ T. Pantev, D. Robbins, T. Vandermeulen

My talk today concerns **decomposition**,
a new notion in quantum field theory (QFT).

Briefly, decomposition is the observation that some local QFTs
are secretly equivalent to
sums of other local QFTs, known as 'universes.'

When this happens, we say the QFT `decomposes.'
Decomposition of the QFT can be applied to give insight
into its properties.

What does it mean for one local QFT to be a sum of other local QFTs?

(Hellerman et al '06)

1) Existence of projection operators

The theory contains topological operators Π_i such that

$$\Pi_i \Pi_j = \delta_{i,j} \Pi_j \qquad \sum_i \Pi_i = 1 \qquad [\Pi_i, \mathcal{O}] = 0$$

Operators Π_i simultaneously diagonalizable; state space = $\mathcal{H} = \oplus_i \mathcal{H}_i$

Correlation functions:

$$\langle \mathcal{O}_1 \cdots \mathcal{O}_m \rangle = \sum_i \langle \Pi_i \mathcal{O}_1 \cdots \mathcal{O}_m \rangle = \sum_i \langle (\Pi_i \mathcal{O}_1) \cdots (\Pi_i \mathcal{O}_m) \rangle = \sum_i \langle \tilde{\mathcal{O}}_1 \cdots \tilde{\mathcal{O}}_m \rangle_i$$

2) Partition functions decompose

$$Z = \sum_{\text{states}} \exp(-\beta H) = \sum_i \sum \exp(-\beta H_i) = \sum_i Z_i$$

(on a connected spacetime)

When does this happen?

There are many examples of decomposition !

Finite gauge theories in 2d (orbifolds): we'll see examples later. (T Pantev, ES '05; D Robbins, ES, T Vandermeulen '21)

Common thread: a subgroup of the gauge group acts trivially.

Example: If $K \subset \text{center}(\Gamma) \subset \Gamma$ acts trivially, then $[X/\Gamma] = \coprod_{\text{irreps } K} [X/(\Gamma/K)]_{\hat{\omega}}$

Gauge theories:

- 2d $U(1)$ gauge theory with nonmin' charges = sum of $U(1)$ theories w/ min charges (Hellerman et al '06)
- 2d G gauge theory w/ center-invt matter = sum of $G/Z(G)$ theories w/ discrete theta (ES '14)

Ex: $SU(2)$ theory (w/ center-invt matter) = $SO(3)_+ \coprod SO(3)_-$ (w/ same matter)

- 2d pure G Yang-Mills = sum of trivial QFTs indexed by irreps of G (Nguyen, Tanizaki, Unsal '21) (U(1): Cherman, Jacobson '20)

Ex: pure $SU(2) = \coprod_{\text{irreps } SU(2)}$ (sigma model on pt)

- 4d Yang-Mills w/ restriction to instantons of deg' divisible by k (Tanizaki, Unsal '19)

= union of ordinary 4d Yang-Mills w/ different θ angles

More examples ....

There are many examples of decomposition !

More examples :

TFTs: 2d unitary TFTs w/ semisimple local operator algebras decompose to invertibles

(Implicit in Durhuus, Jonsson '93; Moore, Segal '06)

(Also: Komargodski et al '20, Huang et al 2110.02958)

Examples:

- 2d abelian BF theory at level k = disjoint union of k invertibles (sigma models on pts)

  (Hellerman, ES, 1012.5999)

- 2d G/G model at level k = disjoint union of invertible theories as many as integrable reps of the Kac-Moody algebra

  (Komargodski et al 2008.07567)

- 2d Dijkgraaf-Witten = sum of invertible theories, as many as irreps (In fact, is a special case of orbifolds discussed later in this talk.)

Sigma models on gerbes = disjoint union of sigma models on spaces w/ B fields

Solves tech issue w/ cluster decomposition.

(T Pantev, ES '05)

What do these examples have in common?....

What do the examples have in common?
When is one local QFT a sum of other local QFTs ?

Answer: in d spacetime dimensions,
a theory decomposes when it has a $(d-1)$-form symmetry.

(2d: Hellerman et al '06;
d>2: Tanizaki-Unsal '19, Cherman-Jacobson '20)

Decomposition & higher-form symmetries go hand-in-hand.

Today I'm primarily interested in the case $d=2$,
so get a decomposition if a $(d-1)=1$-form symmetry is present.

What is a 1-form symmetry?

What is a (linearly realized) one-form symmetry in 2d ?

For this talk, *intuitively*, this will be a `group' that exchanges nonperturbative sectors.

Example: G gauge theory or orbifold in which matter/fields invariant under $K \subset G$

(Technically, to talk about a 1-form symmetry, we assume K abelian, but decompositions exist more generally.)

Then, at least for K central, nonperturbative sectors are invariant under

$$(G-\text{bundle}) \mapsto (G-\text{bundle}) \otimes (K-\text{bundle})$$

$$A \mapsto A + A'$$

At least when K central, this is the action of the `group' of K-bundles.

(Technically, is a 2-group, only weakly associative.)

That group is denoted BK or $K^{(1)}$

One-form symmetries can also be seen in algebra of topological local operators, where they are often realized *non*linearly (eg 2d TFTs). (Komargodski et al '20, Huang et al 2110.02958)

Decomposition vs superselection....

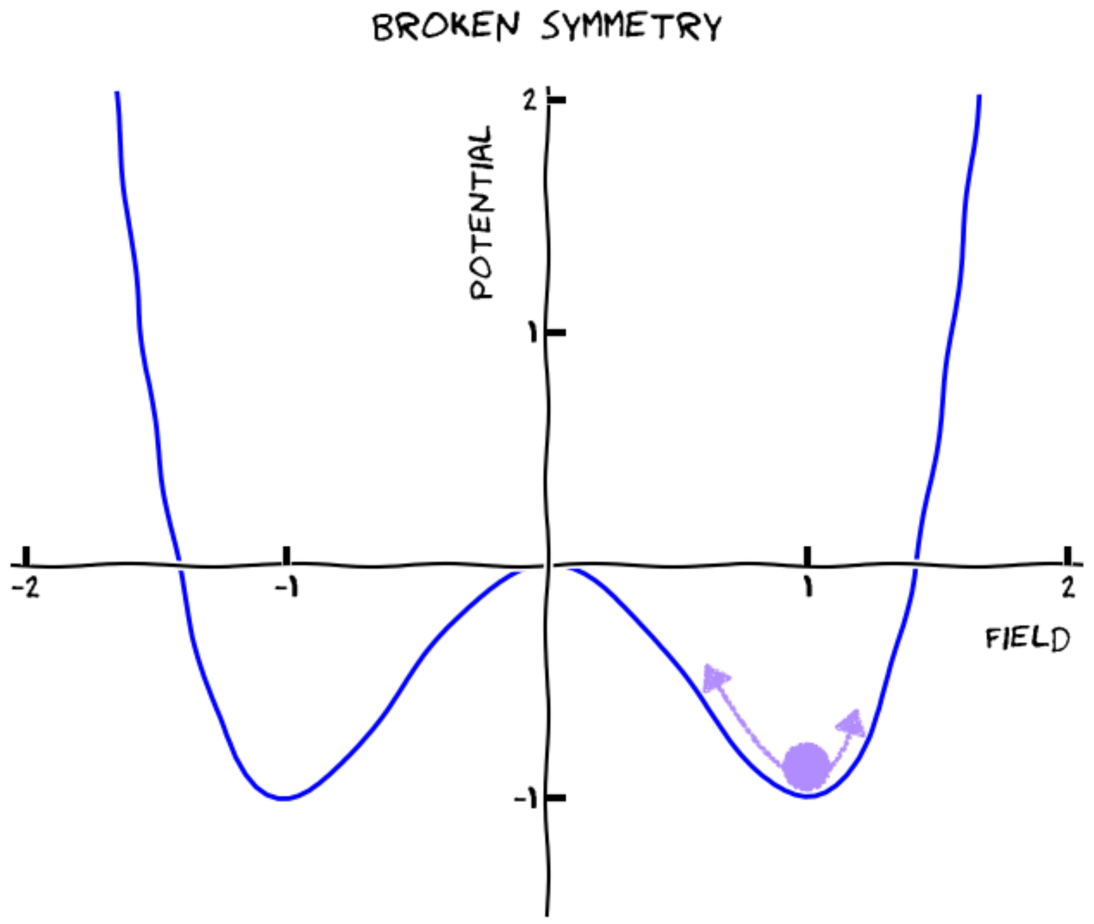
Decomposition ≠ spontaneous symmetry breaking

SSB:

Superselection sectors:

- separated by dynamical domain walls
- only genuinely disjoint in IR
- only one overall QFT

Prototype:

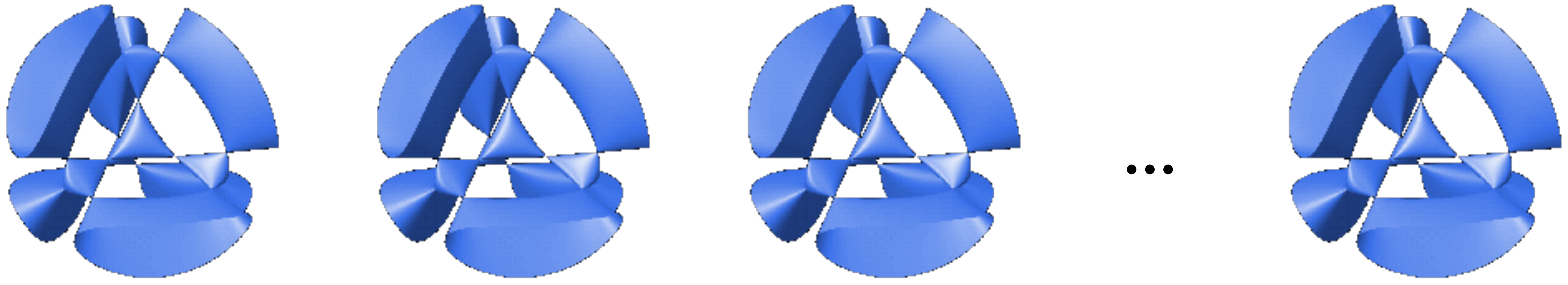


Decomposition:

Universes:

- separated by *non*dynamical domain walls
- disjoint at *all* energy scales
- *multiple* different QFTs present

Prototype:



(see e.g. Tanizaki-Unsal 1912.01033)

Since 2005, decomposition has been checked in many examples in many ways. Examples:

- GLSM's: mirrors, quantum cohomology rings (Coulomb branch) (T Pantev, ES '05; Gu et al '18-'20)
- Orbifolds: partition f'ns, massless spectra, elliptic genera (T Pantev, ES '05; Robbins et al '21)
- Open strings, K theory (Hellerman et al hep-th/0606034)
- Susy gauge theories w/ localization (ES 1404.3986)
- Nonsusy pure Yang-Mills ala Migdal (ES '14; Nguyen, Tanizaki, Unsal '21)
- Adjoint QCD_2 (Komargodski et al '20)
- Numerical checks (lattice gauge thy) (Honda et al '21)
- Versions in d-dim'l theories w/ (d-1)-form symmetries (Tanizaki, Unsal, '19; Cherman, Jacobson '20)

This list is incomplete; apologies to those not listed.

Applications include:

- Sigma models with target stacks & gerbes (T Pantev, ES '05)
- Predictions for Gromov-Witten theory (checked by H-H Tseng, Y Jiang, E Andreini, etc starting '08)
- Nonperturbative constructions of geometries in GLSMs (Caldararu et al 0709.3855, Hori '11, ..., Romo et al '21)
- Elliptic genera (Eager et al '20)
- Anomalies in orbifolds (Robbins et al '21)

The particular QFTs I'm interested in today, which have a decomposition, are (1+1)-dimensional theories with global 1-form symmetries of the following form:

(Pantev, ES '05; Hellerman et al '06)

Symmetry	
1-form	• Gauge theory or orbifold w/ trivially-acting subgroup (<-> non-complete charge spectrum)
$(d-1)$-form	• Theory w/ restriction on instantons
1-form	• Sigma models on gerbes = fiber bundles with fibers = `groups' of 1-form symmetries $G^{(1)} = BG$
$(d-1)$-form	• Algebra of topological local operators

Decomposition (into 'universes') often relates these pictures.

Examples:

restriction on instantons = "multiverse interference effect"

1-form symmetry of QFT = translation symmetry along fibers of gerbe

trivial group action b/c $BG = [\text{point}/G]$

Plan for the rest of the talk:

- **Generalities on gauge theories**
- Specifics in orbifolds
- 3d versions & work in progress

Example: Decomposition in 2d gauge theories

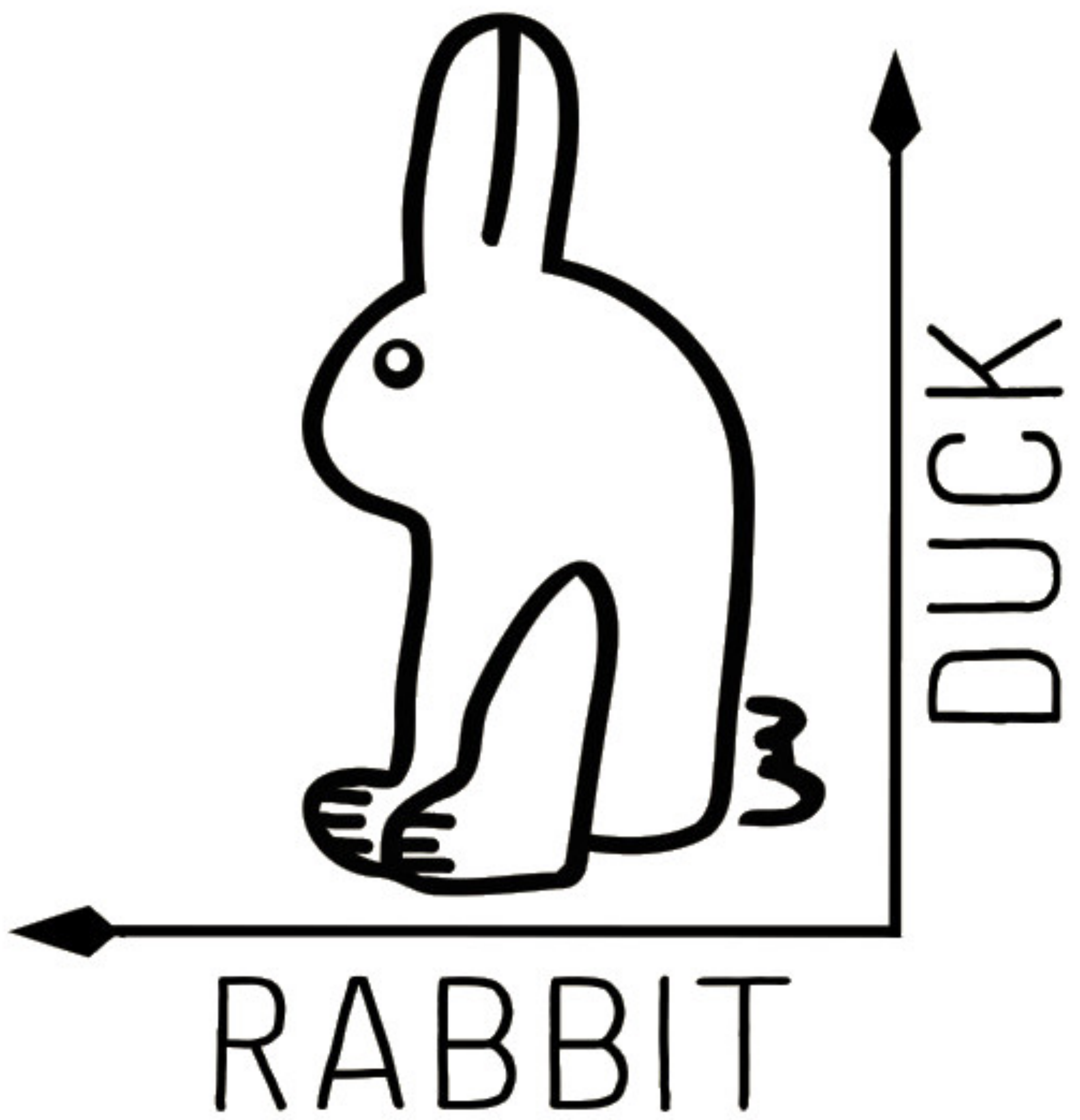
(Hellerman et al '06)

Gauge theory version:

S'pose have $G-$gauge theory, G semisimple, with finite $K \subseteq G$ acting trivially.

For simplicity, assume K is in the center. Has BK 1-form symmetry.

So far, this sounds like just one QFT.



However, I'll outline how, from another perspective, QFTs of this form are also each a disjoint union of other QFTs; they "decompose."

(This will still be somewhat schematic; we'll really dig into details when we get to orbifold examples.)

Example: Decomposition in 2d gauge theories

(Hellerman et al ’06)

Gauge theory version:

S’pose have $G-$gauge theory, G semisimple, with finite $K \subseteq G$ acting trivially.

For simplicity, assume K is in the center. Has BK 1-form symmetry.

Claim this theory decomposes.
Where are the projection operators?

Math understanding:

Briefly, the projection operators (twist fields, Gukov-Witten) correspond to elements of the center of the group algebra $\mathbb{C}[K]$.

Existence of those projectors (idempotents), forming a basis for the center, is ultimately a consequence of Wedderburn’s theorem.

Universes $\longleftrightarrow$ Irreducible representations of K

Partition functions & relation of decomp’ to restrictions on instantons....

Example: Decomposition in 2d gauge theories

(Hellerman et al '06)

Gauge theory version:

S'pose have $G-$gauge theory, G semisimple, with finite $K \subseteq G$ acting trivially.

For simplicity, assume K is in the center. Has BK 1-form symmetry.

Statement of decomposition (in this example):

$$\mathrm{QFT}(G\text{–gauge theory}) = \coprod_{\text{char's } \hat{K}} \mathrm{QFT}\left(G/K\text{–gauge theory w/ discrete theta angles}\right)$$

Example: pure $SU(2)$ gauge theory = sum $SO(3)_+ + SO(3)_-$ pure gauge theories

where $\pm$ denote discrete theta angles (w_2)

Perturbatively, the $SU(2)$, $SO(3)_\pm$ theories are identical
— differences are all nonperturbative.

Example: Decomposition in 2d gauge theories

(Hellerman et al '06)

Gauge theory version:

S'pose have $G-$gauge theory, G semisimple, with finite $K \subseteq G$ acting trivially.

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Example: pure $SU(2)$ gauge theory = sum $SO(3)_+ + SO(3)_-$ pure gauge theories

where $\pm$ denote discrete theta angles (w_2)

$SU(2)$ instantons (bundles) $\subset$ $SO(3)$ instantons (bundles)

The discrete theta angles weight the non-$SU(2)$ $SO(3)$ instantons so as to cancel out of the partition function of the disjoint union.

Summing over the $SO(3)$ theories projects out some instantons, giving the $SU(2)$ theory.

Example: Decomposition in 2d gauge theories

(Hellerman et al '06)

Gauge theory version:

S'pose have $G-$gauge theory, G semisimple, with finite $K \subseteq G$ acting trivially.

For simplicity, assume K is in the center. Has BK 1-form symmetry.

Statement of decomposition (in this example):

$$\text{QFT}(G-\text{gauge theory}) = \coprod_{\text{char's } \hat{K}} \text{QFT}\left(G/K-\text{gauge theory w/ discrete theta angles}\right)$$

Formally, the partition function of the disjoint union can be written

$$Z = \underbrace{\sum_{\theta \in \hat{K}} \int [DA] \exp(-S) \exp\left[\theta \int \omega_2(A)\right]}_{\text{Disjoint union}} = \int [DA] \exp(-S) \overbrace{\left(\sum_{\theta \in \hat{K}} \exp\left[\theta \int \omega_2(A)\right] \right)}^{\text{projection operator}}$$

where we have moved the summation inside the integral.

("multiverse interference" cancels out some sectors)

Example: Decomposition in 2d gauge theories

(Hellerman et al '06)

$$Z = \underbrace{\sum_{\theta \in \hat{K}} \int [DA] \exp(-S) \exp\left[\theta \int \omega_2(A)\right]}_{\text{Disjoint union}} = \int [DA] \exp(-S) \left(\overbrace{\sum_{\theta \in \hat{K}} \exp\left[\theta \int \omega_2(A)\right]}^{\text{projection operator}} \right)$$

Example: Decomposition in 2d gauge theories

(Hellerman et al ’06)

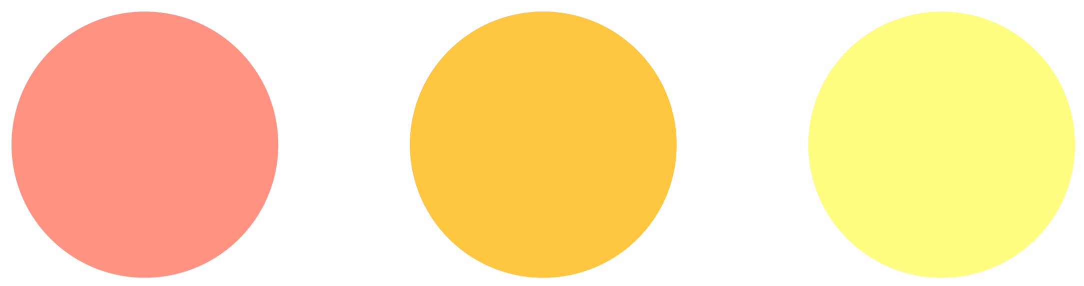
One effect is a projection on nonperturbative sectors:

$$\underbrace{\sum_{\theta \in \hat{K}} \int [DA] \exp(-S) \exp\left[\theta \int \omega_2(A)\right]}_{\text{Disjoint union}} = \int [DA] \exp(-S) \overbrace{\left(\sum_{\theta \in \hat{K}} \exp\left[\theta \int \omega_2(A)\right] \right)}^{\text{projection operator}}$$

Disjoint union of several QFTs / universes = \`One’ QFT with a restriction on nonperturbative sectors = \`multiverse interference’

Schematically,
two theories combine to form a distinct third:

universe
$(SO(3)_+)$



universe
$(SO(3)_-)$

multiverse interference effect
$(SU(2))$

Before going on, let's quickly check these claims for pure $SU(2)$ Yang-Mills in 2d.

The partition function Z, on a Riemann surface of genus g, is

(Migdal, Rusakov)

$$Z(SU(2)) = \sum_R (\dim R)^{2-2g} \exp(-AC_2(R))$$ Sum over all SU(2) reps

$$Z(SO(3)_+) = \sum_R (\dim R)^{2-2g} \exp(-AC_2(R))$$ Sum over all SO(3) reps

(Tachikawa '13)

$$Z(SO(3)_-) = \sum_R (\dim R)^{2-2g} \exp(-AC_2(R))$$ Sum over all SU(2) reps that are not SO(3) reps

Result: $Z(SU(2)) = Z(SO(3)_+) + Z(SO(3)_-)$ as expected.

Another common feature of these theories:
violation of cluster decomposition

As Weinberg taught us years ago,
restricting instantons violates cluster decomposition,
and as we'll see, instanton restriction is a common feature in these theories.

A disjoint union of QFTs also violates cluster decomposition,
but in a trivially controllable fashion.

Lesson: restricting instantons OK,
so long as one has a disjoint union.

(Hellerman, Henriques, T Pantev, ES, M Ando, hep-th/0606034)

Quick note: applications of decomposition in 2d gauge theories

My favorite application: gauged linear sigma models (GLSMs) (Caldararu et al 0709.3855, Hori '11, … …, Romo et al '21)

Can be applied in a Born-Oppenheimer approximation to construct target-space geometries that are branched covers.

Other applications:

- Elliptic genera of pure susy gauge theories (Eager et al '20)

  (to check claims about IR limits)

- Gromov-Witten invariants of stacks & gerbes

  (checked by H-H Tseng, Y Jiang, E Andreini, etc starting '08)

Next: orbifolds

Plan for the rest of the talk:

- Generalities on gauge theories
- **Specifics in orbifolds**
- 3d versions & work in progress

Decomposition in orbifolds in (1+1) dimensions

Let's begin by discussing ordinary orbifolds.

(Versions exist for orbifolds with discrete torsion and quantum symmetries, which have been applied to e.g. Wang-Wen-Witten anomaly resolution, but in this talk I'll focus on basic cases.)

Consider an orbifold $[X/\Gamma]$, where $K \subset \Gamma$ acts trivially.

$$1 \longrightarrow K \longrightarrow \Gamma \longrightarrow G \longrightarrow 1 \qquad (K, \Gamma, G \text{ finite})$$

For simplicity, assume K central.

Decomposition implies (Hellerman et al '06)

$$\mathrm{QFT}\left([X/\Gamma]\right) = \mathrm{QFT}\left(\coprod_{\hat{K}} [X/G]_{\hat{\omega}} \right)$$

$\hat{K}$ = set of iso classes of irreps of K

$\hat{\omega}$ = phases called "discrete torsion".

$$= \mathrm{Image}\left(H^2(G,K) \xrightarrow{\theta \in \hat{K}} H^2(G,U(1)) \right)$$

Note similar to gauge theories:
$SU(2) = SO(3)_+ + SO(3)_-$

Decomposition in orbifolds in (1+1) dimensions

Consider an orbifold $[X/\Gamma]$, where $K \subset \Gamma$ acts trivially.

$$1 \longrightarrow K \longrightarrow \Gamma \longrightarrow G \longrightarrow 1$$ (assume K central)

Decomposition implies

(Hellerman et al '06)

$$\mathrm{QFT}\left([X/\Gamma]\right) = \mathrm{QFT}\left(\coprod_{\hat{K}} \left[X/G\right]_{\hat{\omega}} \right)$$

$\hat{K}$ = set of iso classes of irreps of K

Projectors: For $R \in \hat{K}$, we have the projector

$$\Pi_R = \sum_i \frac{\dim R_i}{|K|} \sum_{k \in K} \chi_{R_i}(k^{-1}) \tau_k$$

(Wedderburn's theorem for center of group algebra)

which obey $\Pi_R \Pi_S = \delta_{R,S} \Pi_R, \quad \sum_R \Pi_R = 1$

To make this more concrete, let's walk through an example, where everything can be made completely explicit.

Example: Orbifold $[X/D_4]$ in which the $\mathbb{Z}_2$ center acts trivially.

(T Pantev, ES '05)

— has $B\mathbb{Z}_2$ (1-form) symmetry

$D_4/\mathbb{Z}_2 = \mathbb{Z}_2 \times \mathbb{Z}_2$ so this is closely related to a $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold

Decomposition predicts

$$\mathrm{QFT}\left([X/D_4]\right) = \mathrm{QFT}\left([X/\mathbb{Z}_2 \times \mathbb{Z}_2]_{\text{w/o d.t.}}\right) \coprod \mathrm{QFT}\left([X/\mathbb{Z}_2 \times \mathbb{Z}_2]_{\text{d.t.}}\right)$$

Let's check this explicitly....

Example, cont'd

$$\mathrm{QFT}\left([X/D_4]\right) \; = \; \mathrm{QFT}\left([X/\mathbb{Z}_2 \times \mathbb{Z}_2]_{\text{w/o d.t.}}\right) \coprod \mathrm{QFT}\left([X/\mathbb{Z}_2 \times \mathbb{Z}_2]_{\text{d.t.}}\right)$$

At the level of operators, one reason for this is that the theory admits projection operators:

Let $\hat{z}$ denote the (dim 0) twist field associated to the trivially-acting $\mathbb{Z}_2$:

$\hat{z}$ obeys $\hat{z}^2 \; = \; 1.$

Using that relation, we form projection operators:

$$\Pi_{\pm} \; = \; \frac{1}{2}(1 \pm \hat{z})$$

(= specialization of formula given earlier)

$$\Pi_{\pm}^2 = \Pi_{\pm} \qquad \Pi_{\pm}\Pi_{\mp} = 0 \qquad \Pi_{+} + \Pi_{-} = 1$$

Next: compare partition functions....

Example, cont'd

Compute the partition function of $[X/D_4]$

(T Pantev, ES '05)

$$D_4 = \{1, z, a, b, az, bz, ab, ba = abz\}$$

where z generates the $\mathbb{Z}_2$ center.

Take the (1+1)-dim'l spacetime to be T^2.

The partition function of any orbifold $[X/\Gamma]$ on T^2 is

$$Z_{T^2}([X/\Gamma]) = \frac{1}{|\Gamma|} \sum_{gh=hg} Z_{g,h} \quad \text{where } Z_{g,h} = \left(g \underset{h}{\blacksquare} \longrightarrow X \right)$$

("twisted sectors")

(Think of $Z_{g,h}$ as sigma model to X with branch cuts g, h.)

We're going to see that

$$Z_{T^2}\left([X/D_4]\right) = Z_{T^2}\left([X/\mathbb{Z}_2 \times \mathbb{Z}_2]\right) + Z_{T^2}\left([X/\mathbb{Z}_2 \times \mathbb{Z}_2]_{\text{d.t.}}\right)$$

Example, cont'd

Compute the partition function of $[X/D_4]$

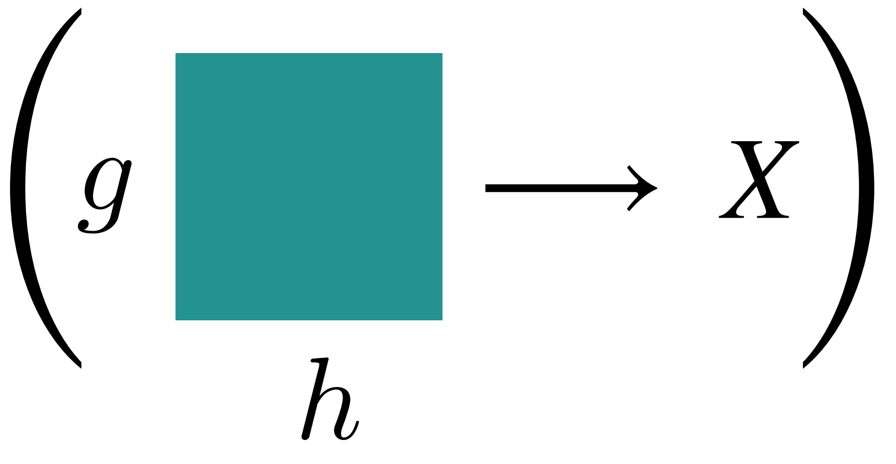
(T Pantev, ES '05)

$$D_4 = \{1, z, a, b, az, bz, ab, ba = abz\}$$

where z generates the $\mathbb{Z}_2$ center.

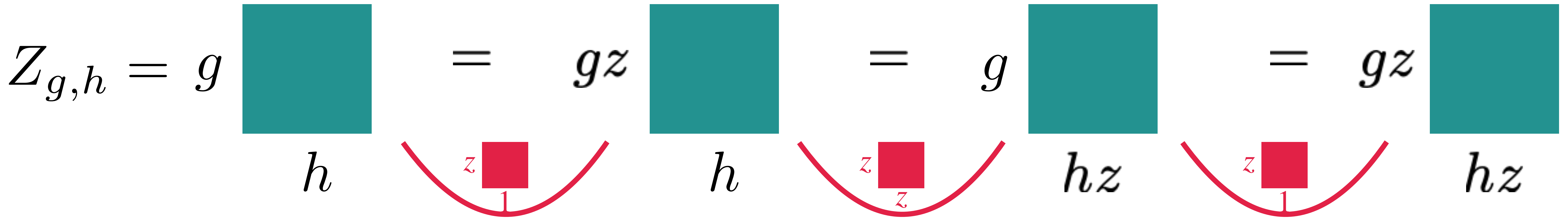
$$D_4/\mathbb{Z}_2 = \mathbb{Z}_2 \times \mathbb{Z}_2 = \{1, \overline{a}, \overline{b}, \overline{ab}\}$$ where $\overline{a} = \{a, az\}$ etc

$$Z_{T^2}\left([X/D_4]\right) = \frac{1}{|D_4|} \sum_{g,h \in D_4,\ gh=hg} Z_{g,h}$$ where $Z_{g,h}$ =



Since z acts trivially,

$Z_{g,h}$ is symmetric under multiplication by z



This is the $B\mathbb{Z}_2$ 1-form symmetry.

Example, cont'd

Compute the partition function of $[X/D_4]$ (T Pantev, ES '05)

$$D_4 = \{1, z, a, b, az, bz, ab, ba = abz\}$$

where z generates the $\mathbb{Z}_2$ center.

$$D_4/\mathbb{Z}_2 = \mathbb{Z}_2 \times \mathbb{Z}_2 = \{1, \overline{a}, \overline{b}, \overline{ab}\}$$ where $\overline{a} = \{a, az\}$ etc

$$Z_{T^2}([X/D_4]) = \frac{1}{|D_4|} \sum_{g,h \in D_4,\ gh=hg} Z_{g,h}$$ where $Z_{g,h} = \left(g \underset{h}{\blacksquare} \longrightarrow X \right)$

Each D_4 twisted sector ($Z_{g,h}$) that appears is the same as a $D_4/\mathbb{Z}_2 = \mathbb{Z}_2 \times \mathbb{Z}_2$ twisted sector,

appearing with multiplicity $|\mathbb{Z}_2|^2 = 4$,

except for the sectors $\overline{a} \underset{\overline{b}}{\blacksquare}$ $\overline{a} \underset{\overline{ab}}{\blacksquare}$ $\overline{b} \underset{\overline{ab}}{\blacksquare}$ which do **not** appear.

Restriction on nonperturbative sectors

Example, cont'd

Compute the partition function of $[X/D_4]$

(T Pantev, ES '05)

$$Z_{T^2}\left([X/D_4]\right) = \frac{|\mathbb{Z}_2 \times \mathbb{Z}_2|}{|D_4|} |\mathbb{Z}_2|^2 \left(Z_{T^2}\left([X/\mathbb{Z}_2 \times \mathbb{Z}_2]\right) - (\text{some twisted sectors}) \right)$$

$$= 2\left(Z_{T^2}\left([X/\mathbb{Z}_2 \times \mathbb{Z}_2]\right) - (\text{some twisted sectors}) \right)$$

Different theory than $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold

Physics knows when we gauge even a trivially-acting group!

Example, cont'd

Compute the partition function of $[X/D_4]$

(T Pantev, ES '05)

$$Z_{T^2}\left([X/D_4]\right) = \frac{|\mathbb{Z}_2 \times \mathbb{Z}_2|}{|D_4|} |\mathbb{Z}_2|^2 \left(Z_{T^2}\left([X/\mathbb{Z}_2 \times \mathbb{Z}_2]\right) - \left(\text{some twisted sectors}\right)\right)$$

$$= 2\left(Z_{T^2}\left([X/\mathbb{Z}_2 \times \mathbb{Z}_2]\right) - \left(\text{some twisted sectors}\right)\right)$$

Fact: given any one partition function $Z_{T^2}([X/G]) = \frac{1}{|G|} \sum_{gh=hg} Z_{g,h}$

we can multiply in $SL(2,\mathbb{Z})$-invariant phases $\epsilon(g,h)$

to get another consistent partition function (for a different theory)

$$Z' = \frac{1}{|G|} \sum_{gh=hg} \epsilon(g,h)\, Z_{g,h}$$

There is a universal choice of such phases, determined by elements of $H^2(G, U(1))$

This is called "discrete torsion."

Example, cont'd

Compute the partition function of $[X/D_4]$

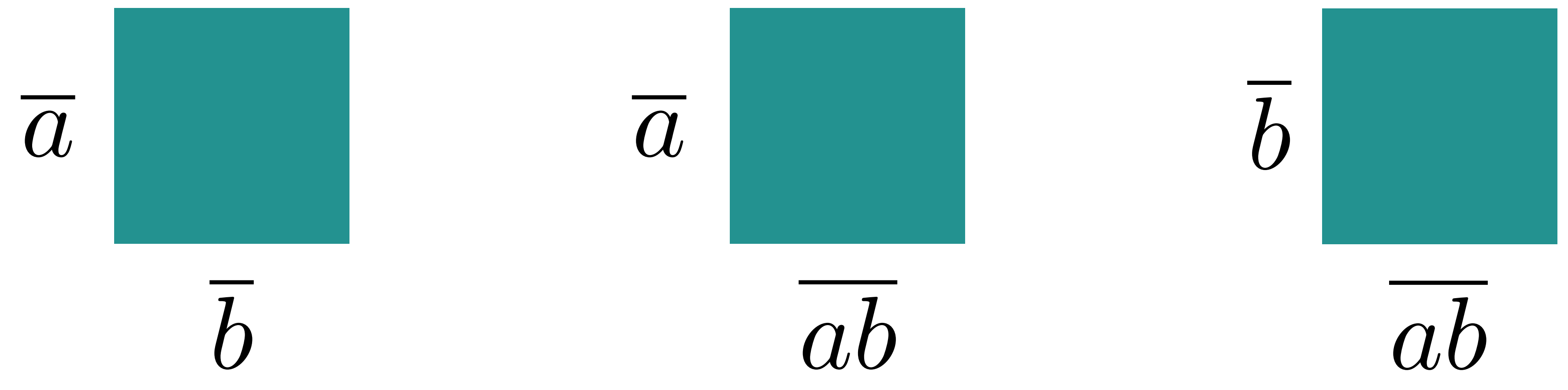
(T Pantev, ES '05)

$$Z_{T^2}\left([X/D_4]\right) = \frac{|\mathbb{Z}_2\times\mathbb{Z}_2|}{|D_4|}|\mathbb{Z}_2|^2\left(Z_{T^2}\left([X/\mathbb{Z}_2\times\mathbb{Z}_2]\right) - \left(\text{some twisted sectors}\right)\right)$$

$$= 2\left(Z_{T^2}\left([X/\mathbb{Z}_2\times\mathbb{Z}_2]\right) - \left(\text{some twisted sectors}\right)\right)$$

In a $\mathbb{Z}_2\times\mathbb{Z}_2$ orbifold, discrete torsion $\in H^2(\mathbb{Z}_2\times\mathbb{Z}_2, U(1)) = \mathbb{Z}_2$,

and the nontrivial element acts as a sign on the twisted sectors



the same sectors which were omitted above.

$$Z_{T^2}\left([X/D_4]\right) = Z_{T^2}\left([X/\mathbb{Z}_2\times\mathbb{Z}_2]_{\text{w/o d.t.}}\right) + Z_{T^2}\left([X/\mathbb{Z}_2\times\mathbb{Z}_2]_{\text{d.t.}}\right)$$

Adding the universes projects out some sectors — interference effect.

Example, cont'd

Compute the partition function of $[X/D_4]$

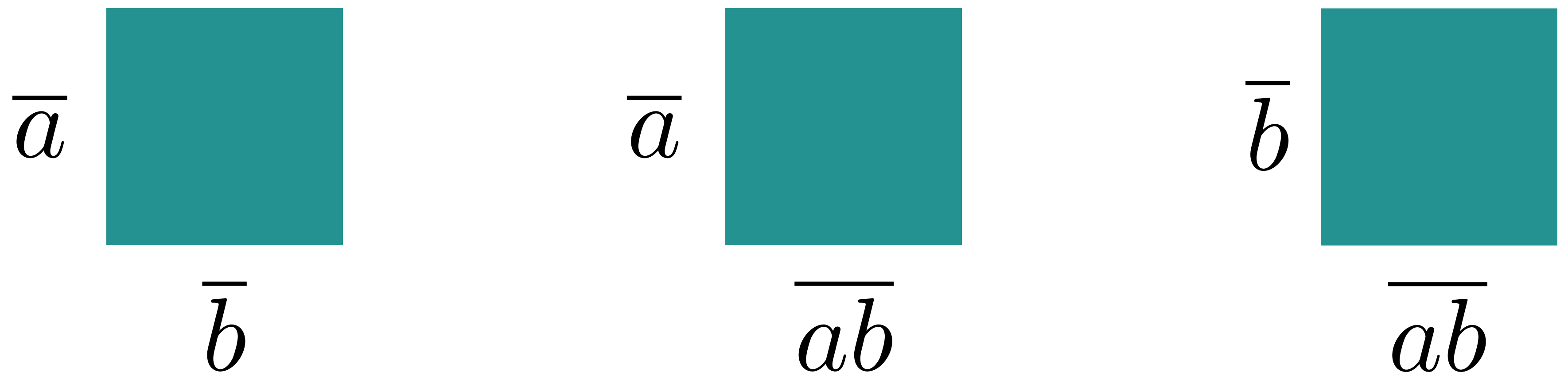
(T Pantev, ES '05)

$$Z_{T^2}\left([X/D_4]\right) \;=\; \frac{|\mathbb{Z}_2 \times \mathbb{Z}_2|}{|D_4|}\,|\mathbb{Z}_2|^2 \left(Z_{T^2}\left([X/\mathbb{Z}_2 \times \mathbb{Z}_2]\right) \;-\; (\text{some twisted sectors}) \right)$$

$$= \; 2\left(Z_{T^2}\left([X/\mathbb{Z}_2 \times \mathbb{Z}_2]\right) \;-\; (\text{some twisted sectors}) \right)$$

Discrete torsion is $H^2(\mathbb{Z}_2 \times \mathbb{Z}_2, U(1)) = \mathbb{Z}_2$,

and acts as a sign on the twisted sectors



which were omitted above.

$$Z_{T^2}\left([X/D_4]\right) \;=\; Z_{T^2}\left([X/\mathbb{Z}_2 \times \mathbb{Z}_2]_{\text{w/o d.t.}}\right) \;+\; Z_{T^2}\left([X/\mathbb{Z}_2 \times \mathbb{Z}_2]_{\text{d.t.}}\right)$$

Matches prediction of decomposition

$$\text{QFT}\left([X/D_4]\right) \;=\; \text{QFT}\left([X/\mathbb{Z}_2 \times \mathbb{Z}_2]_{\text{w/o d.t.}}\right) \coprod \text{QFT}\left([X/\mathbb{Z}_2 \times \mathbb{Z}_2]_{\text{d.t.}}\right)$$

Example, cont'd

$$Z_{T^2}\left([X/D_4]\right) \; = \; Z_{T^2}\left([X/\mathbb{Z}_2\times\mathbb{Z}_2]_{\rm w/o\ d.t.}\right) \; + \; Z_{T^2}\left([X/\mathbb{Z}_2\times\mathbb{Z}_2]_{\rm d.t.}\right)$$

Matches prediction of decomposition

$${\rm QFT}\left([X/D_4]\right) \; = \; {\rm QFT}\left([X/\mathbb{Z}_2\times\mathbb{Z}_2]_{\rm w/o\ d.t.}\right) \coprod {\rm QFT}\left([X/\mathbb{Z}_2\times\mathbb{Z}_2]_{\rm d.t.}\right)$$

The computation above demonstrated that the partition function on T^2 has the form predicted by decomposition.
The same is also true of partition functions at higher genus
— just more combinatorics.
(see hep-th/0606034, section 5.2 for details)

Only slightly novel aspect: in gen'l, one finds dilaton shifts, which mostly I'll suppress in this talk.

Example, cont'd

Massless states of $[X/D_4]$ for $X = T^6$ (T Pantev, ES '05)

Massless states of $[T^6/D_4]$

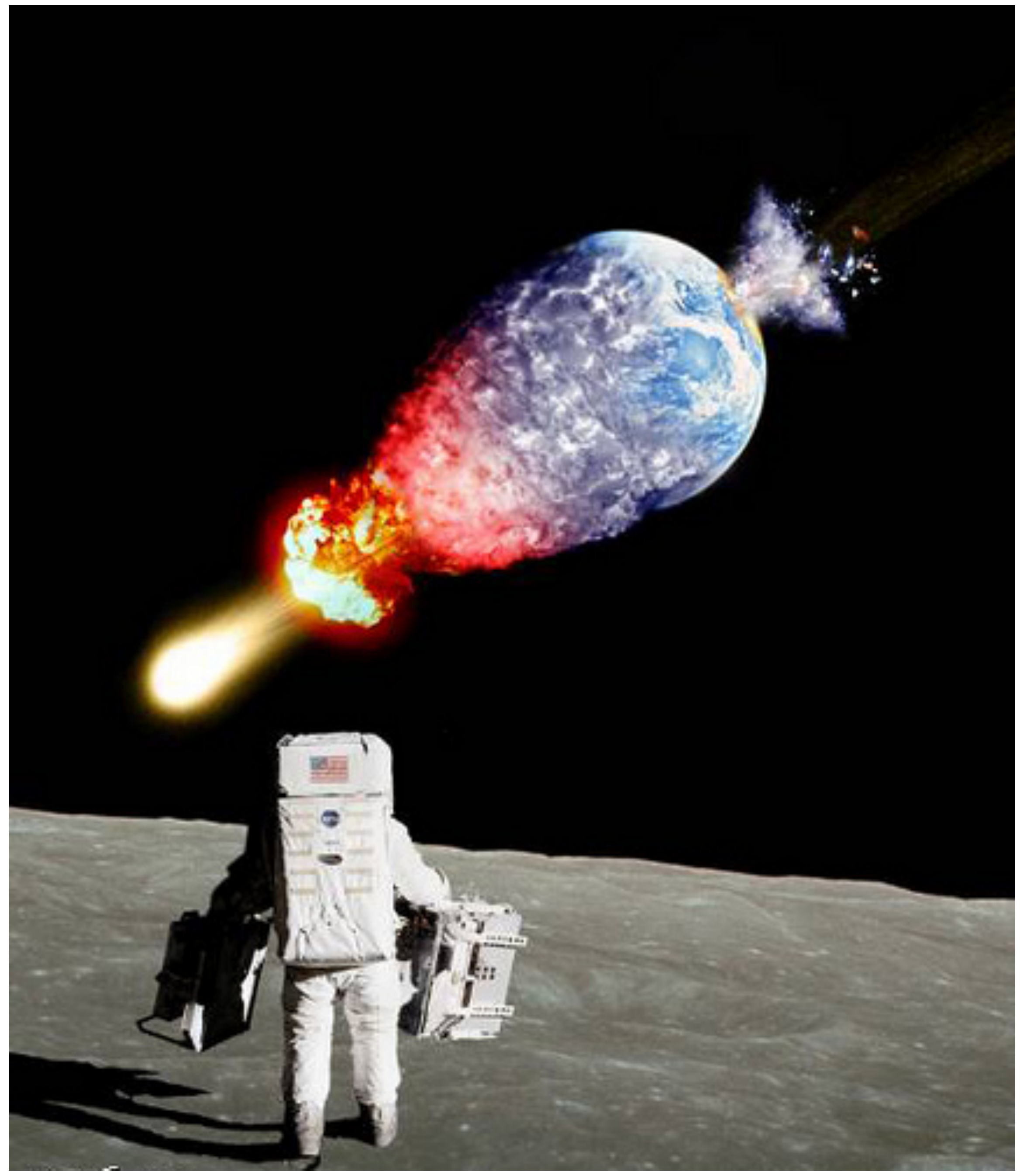
```
         2
      0     0
   0     54    0
2    54    54    2
   0     54    0
      0     0
         2
```

Signals mult' components / cluster decomp' violation

If we didn't know about decomposition, the 2's in the corners would be a problem...

A big problem!

They signal a violation of cluster decomposition, the same axiom that's violated by restricting instantons.



Ordinarily, I'd assume that the computation was wrong.

However, decomposition saves the day....

Example, cont'd

Massless states of $[X/D_4]$ for $X = T^6$ (T Pantev, ES '05)

Massless states of $[T^6/D_4]$

$$
\begin{array}{ccccccc}
 & & & 2 & & & \\
 & & 0 & & 0 & & \\
 & 0 & & 54 & & 0 & \\
2 & & 54 & & 54 & & 2 \\
 & 0 & & 54 & & 0 & \\
 & & 0 & & 0 & & \\
 & & & 2 & & &
\end{array}
\quad = \quad
\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 0 & & 0 & & \\
 & 0 & & 51 & & 0 & \\
1 & & 3 & & 3 & & 1 \\
 & 0 & & 51 & & 0 & \\
 & & 0 & & 0 & & \\
 & & & 1 & & &
\end{array}
\quad + \quad
\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 0 & & 0 & & \\
 & 0 & & 3 & & 0 & \\
1 & & 51 & & 51 & & 1 \\
 & 0 & & 3 & & 0 & \\
 & & 0 & & 0 & & \\
 & & & 1 & & &
\end{array}
$$

Signals mult' components / cluster decomp' violation

spectrum of $\mathbb{Z}_2 \times \mathbb{Z}_2$ orb' w/o d.t.

spectrum of $\mathbb{Z}_2 \times \mathbb{Z}_2$ orb' w/ d.t.

matching the prediction of decomposition

$$\mathrm{CFT}\left([X/D_4]\right) = \mathrm{CFT}\left([X/\mathbb{Z}_2 \times \mathbb{Z}_2]_{\mathrm{w/o\,d.t.}}\right) \coprod \mathrm{CFT}\left([X/\mathbb{Z}_2 \times \mathbb{Z}_2]_{\mathrm{d.t.}}\right)$$

This computation was not a one-off, but in fact verifies a prediction in Hellerman et al '06 regarding QFTs in (1+1)-dims with 1-form symmetry.

Another example: Triv'ly acting subgroup **not** in center

Consider $[X/\mathbb{H}]$, $\mathbb{H}$ = eight-element gp of unit quaternions, where $\langle i \rangle = \mathbb{Z}_4 \subset \mathbb{H}$ acts trivially.

Decomposition predicts

$$\mathrm{QFT}\left([X/\Gamma]\right) = \mathrm{QFT}\left(\left[\frac{X \times \hat{K}}{G} \right]_{\hat{\omega}} \right)$$

(Hellerman et al '06)

where $\hat{K}$ = irreps of K

$\hat{\omega}$ = discrete torsion on universes

Here, $G = \mathbb{H}/\langle i \rangle = \mathbb{Z}_2$ acts nontriv'ly on $\hat{K} = \mathbb{Z}_4$, interchanging 2 elements,

so $$\mathrm{QFT}\left([X/\mathbb{H}]\right) = \mathrm{QFT}\left(X \coprod [X/\mathbb{Z}_2] \coprod [X/\mathbb{Z}_2] \right)$$

(Hellerman et al, hep-th/0606034, sect. 5.4)

— different universes; $X \neq [X/\mathbb{Z}_2]$

— easily checked

Quick note: applications of decomposition in 2d orbifolds

One recent application was to understand Wang-Wen-Witten's work on anomaly resolution. (Robbins et al '21)

Briefly, given an orbifold $[X/G]$ with a gauge anomaly,
Wang-Wen-Witten abstractly construct a related orbifold $[X/\Gamma]_B$,
with a trivially-acting $K \subset \Gamma$,
which in principle is anomaly free.

However, it was shown using decomposition in (Robbins et al '21) that

$$[X/\Gamma]_B \; = \; \coprod \, [X / \text{anomaly-free subgp of } G]$$

which gives a simple way to understand why WWW's procedure works.

Next: 3d

Plan for the rest of the talk:

- Generalities on gauge theories
- Specifics in orbifolds
- **3d versions & work in progress**

Three-dimensional examples

Let's construct an example of a decomposition in 3d.

We need a theory with a global 2-form symmetry.

One way to get that is by gauging a trivially-acting one-form symmetry,
by which we mean, for example, line operators have no braiding.

Three-dimensional examples

(T Pantev, D Robbins, T Vandermeulen, ES 2204.13708)

Example: Consider an orbifold $[X/\Gamma]$ where

$$1 \longrightarrow BK \longrightarrow \Gamma \longrightarrow G \longrightarrow 1 \qquad [\omega] \in H^3(G,K)$$

G, K finite; K abelian; BK acts trivially.

Since BK acts trivially, this theory should have a global 2-form symmetry, & so decompose.

Let's see that explicitly.

Projectors: Projectors are constructed from monopole operators associated to the BK, which generate K-gerbes on surrounding S^2's.

For example, if $K = \mathbb{Z}_k$, then as $\mathbb{Z}_k$-gerbes on S^2 have one generator, there is one generating monopole operator, call it $\hat{z}$, with the property $\hat{z}^k = 1$.

$$\Pi_n = \frac{1}{k}\sum_{m=0}^{k-1} \xi^{mn} \hat{z}^m \qquad \text{where } \xi = \exp(2\pi i/k)$$

Three-dimensional examples

(T Pantev, D Robbins, T Vandermeulen, ES 2204.13708)

Example: Consider an orbifold $[X/\Gamma]$ where

$$1 \longrightarrow BK \longrightarrow \Gamma \longrightarrow G \longrightarrow 1 \qquad [\omega] \in H^3(G, K)$$

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Since BK acts trivially, this theory should have a global 2-form symmetry, & so decompose.

We find:

$$\mathrm{QFT}\left([X/\Gamma]\right) = \coprod_{\rho \in \hat{K}} \mathrm{QFT}\left([X/G]_{\rho \circ \epsilon} \right)$$

(closely analogous to 2d orbifolds with trivially-acting K)

Three-dimensional examples

(T Pantev, D Robbins, T Vandermeulen, ES 2204.13708)

Example: Consider an orbifold $[X/\Gamma]$ where

$$1 \longrightarrow BK \longrightarrow \Gamma \longrightarrow G \longrightarrow 1 \qquad [\omega] \in H^3(G,K)$$

G, K finite; K abelian; BK acts trivially. Claim $[X/\Gamma]$ decomposes.

Partition function:

In general terms, the path integral for the orbifold $[X/\Gamma]$ involves a sum over

- principal Γ-bundles E over the 3-manifold M_3
- Maps $E \to X$

just like an ordinary orbifold.

Also, since BK acts trivially, the twisted sectors will be those of a G orbifold.

However, those G-twisted sectors are restricted….

Three-dimensional examples

(T Pantev, D Robbins, T Vandermeulen, ES 2204.13708)

Example: Consider an orbifold $[X/\Gamma]$ where

$$1 \longrightarrow BK \longrightarrow \Gamma \longrightarrow G \longrightarrow 1 \qquad [\omega] \in H^3(G,K)$$

G, K finite; K abelian; BK acts trivially. Claim $[X/\Gamma]$ decomposes.

Partition function:

On T^3, the sum over Γ-twisted sectors maps to a sum over G-twisted sectors such that

$$\epsilon(g_1,g_2,g_3) = \frac{\omega(g_1,g_2,g_3)}{\omega(g_2,g_1,g_3)} \frac{\omega(g_3,g_1,g_2)}{\omega(g_1,g_3,g_2)} \frac{\omega(g_2,g_3,g_1)}{\omega(g_3,g_2,g_1)} = 1 \in K$$

— restriction on nonperturbative sectors

We can implement that restriction by inserting a projection operator

$$\Pi = \frac{1}{|K|} \sum_{\rho \in \hat{K}} \rho \circ \epsilon$$

Partition function....

(T Pantev, D Robbins, T Vandermeulen, ES 2204.13708)

Three-dimensional examples

Example: Consider an orbifold $[X/\Gamma]$ where

$$1 \longrightarrow BK \longrightarrow \Gamma \longrightarrow G \longrightarrow 1 \qquad [\omega] \in H^3(G,K)$$

G, K finite; K abelian; BK acts trivially. Claim $[X/\Gamma]$ decomposes.

Partition function on T^3:

$$Z_{T^3}([X/\Gamma]) = \frac{|H^0(T^3,K)|}{|H^1(T^3,K)|}\frac{1}{|H^0(T^3,G)|}\sum_{z_{1-3}\in K}\sum_{g_{1-3}\in G} \Pi\, Z(g_1,g_2,g_3)$$

$$= \frac{1}{|K|^2|G|}|K|^3 \sum_{g_{1-3}\in G}\frac{1}{|K|}\sum_{\rho\in\hat{K}}(\rho\circ\epsilon)(g_1,g_2,g_3)\, Z(g_1,g_2,g_3)$$

$$= \sum_{\rho\in\hat{K}} Z_{T^3}\left([X/G]_{\rho\circ\epsilon}\right)$$

where $\rho \circ \epsilon$ defines C-field-analogue of discrete torsion

consistent with $$\mathrm{QFT}\,([X/\Gamma]) = \coprod_{\rho\in\hat{K}} \mathrm{QFT}\left([X/G]_{\rho\circ\epsilon}\right)$$ Decomposition

Three-dimensional examples

(T Pantev, D Robbins, T Vandermeulen, ES 2204.13708)

Example: Consider an orbifold $[X/\Gamma]$ where

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Partition function on T^3:

$$Z_{T^3}([X/\Gamma]) = \sum_{\rho \in \hat{K}} Z_{T^3}\left([X/G]_{\rho \circ \epsilon} \right)$$

where $\rho \circ \epsilon$ defines C-field-analogue of discrete torsion

consistent with

$$\mathrm{QFT}\left([X/\Gamma]\right) = \coprod_{\rho \in \hat{K}} \mathrm{QFT}\left([X/G]_{\rho \circ \epsilon} \right)$$

Decomposition

Similar results arise on other 3-manifolds.

Three-dimensional examples

Work in progress

Example: Chern-Simons theories

Chern-Simons theories are particularly interesting for these ideas.

For example, classically AdS_3 is Chern-Simons for $SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$,
so understanding decomposition in Chern-Simons theories
may give toy models of issues in gravity theories
such as Marolf-Maxfield factorization.

So, what's the decomposition in Chern-Simons ?

Three-dimensional examples

Work in progress

Example: Chern-Simons theories

Consider Chern-Simons(H) / BA for A finite & abelian.

There is an associated `crossed module'

$$1 \longrightarrow K\,(=\ker d) \longrightarrow A \xrightarrow{d} H \longrightarrow G\,(=H/\mathrm{im}\,d) \longrightarrow 1$$

Similar remarks apply: only restricted G bundles can appear.

To implement that restriction, must sum over universes....

Conjecture:

$$\text{Chern-Simons}(H)\,/\,BA = \coprod_{\rho \in \hat{K}} \text{Chern-Simons}(G)_{\omega(\rho)}$$

Decomposition

Three-dimensional examples

Work in progress

Example: Chern-Simons theories

Consider Chern-Simons(H) / BA for A finite & abelian.

Conjecture: $$\text{Chern-Simons}(H) \,/\, BA = \coprod_{\rho \in \hat{K}} \text{Chern-Simons}(G)_{\omega(\rho)}$$

Example: Chern-Simons($SU(2)$) / $B\mathbb{Z}_2$ where the $B\mathbb{Z}_2$ acts via the center

$$1 \longrightarrow K\,(=1) \longrightarrow \mathbb{Z}_2 \xrightarrow{d} SU(2) \longrightarrow SO(3)\,(=SU(2)/\operatorname{im} d) \longrightarrow 1$$

so predict

$$\text{Chern-Simons}(SU(2)) \,/\, B\mathbb{Z}_2 = \text{Chern-Simons}(SO(3))$$

which is a standard result.

Three-dimensional examples

Work in progress

Example: Chern-Simons theories

Consider Chern-Simons(H) / BA for A finite & abelian.

Conjecture:

$$\text{Chern-Simons}(H) \,/\, BA = \coprod_{\rho \in \hat{K}} \text{Chern-Simons}(G)_{\omega(\rho)}$$

Example: Chern-Simons($SU(2)$) / $B\mathbb{Z}_4$ where the $B\mathbb{Z}_4$ maps to the center

$$1 \longrightarrow K\,(=\mathbb{Z}_2) \longrightarrow \mathbb{Z}_4 \xrightarrow{d} SU(2) \longrightarrow SO(3)\,(=SU(2)/\mathrm{im}\,d) \longrightarrow 1$$

so predict

$$\text{Chern-Simons}(SU(2)) \,/\, B\mathbb{Z}_4 = \coprod_{\rho \in \hat{\mathbb{Z}}_2} \text{Chern-Simons}(SO(3))_{\omega(\rho)}$$

where here ω couples to third Stiefel-Whitney class.

Three-dimensional examples

Work in progress

Example: Chern-Simons theories

Consider Chern-Simons(H) / BA for A finite & abelian.

Conjecture:

$$\text{Chern-Simons}(H) \,/\, BA \;=\; \coprod_{\rho \in \hat{K}} \text{Chern-Simons}(G)_{\omega(\rho)}$$

How to check?

For example, boundaries. Above becomes

$$\text{WZW}(H)/A \;=\; \coprod_{\rho \in \hat{K}} \text{WZW}(G)_{\theta(\rho)}$$

where the boundary discrete theta angle related to bulk via transgression.

Can show, in fact, boundary discrete theta angle = discrete torsion,
and the predicted boundary decomposition = standard 2d orbifold decomposition.

Summary:

- Generalities on gauge theories
- Specifics in orbifolds
- 3d versions & work in progress

Fun features of decomposition:

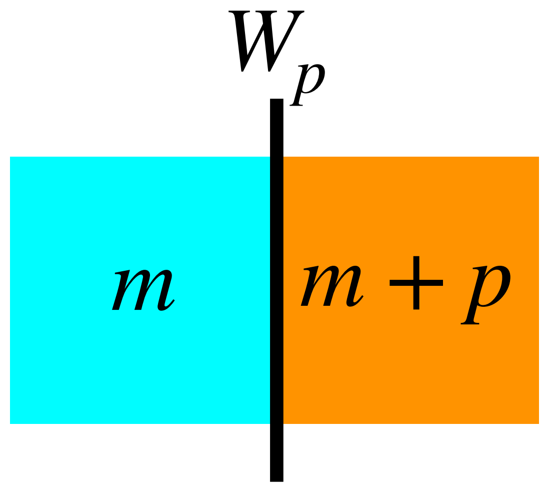
Multiverse interference effects

Ex: 2d $SU(2)$ gauge theory w/ center-invariant matter = $SO(3)_+ + SO(3)_-$

Summing over the two universes ($SO(3)$ gauge theories)
cancels out $SO(3)$ bundles which don't arise from $SU(2)$.

Wilson lines = defects between universes

Ex: 2d abelian BF theory at level k



Projectors: $\Pi_m = \frac{1}{k}\sum_{n=0}^{k-1} \xi^{nm}\mathcal{O}_n \qquad \xi = \exp(2\pi i/k)$

Clock-shift commutation relations: $\mathcal{O}_p W_q = \xi^{pq} W_q \mathcal{O}_p \quad \Leftrightarrow \quad \Pi_m W_p = W_p \Pi_{m+p \mod k}$

Wormholes between universes (GLSMs: Caldararu et al, 0709.3855)

Ex: U(1) susy gauge theory in 2d: 2 chirals p charge 2, 4 chirals ϕ charge -1, $W = \sum_{ij} \phi_i \phi_j A^{ij}(p)$

Describes double cover of $\mathbb{P}^1$ (sheets are universes), linked over locus where ϕ massless — Euclidean wormhole

Conclusions

Decomposition: `one' local QFT is secretly several

Decomposition appears in $(n+1)-$dimensional theories with $n-$form symmetries.

(I've mostly focused on examples in 2d, but examples exist in other dim's too.)

Thank you for your time!