$T\bar{T}$ and holography

Wei Song

Tsinghua University

Luis Apolo and WS, 1806.10127, 1907.03745, 2111.02243

Luis Apolo, Stephane Detournay and WS, 1911.12359

Luis Apolo, Penxiang Hao, Wenxin Lai and WS, WIP

BIMSA, June 16, 2022

Quantum gravity in the real world?

AdS $\Longleftrightarrow$ CFT

non-AdS geometries in our real world:

-Minkowski

-de Sitter

-Kerr

irrelevant deformations

- The $T\bar{T}$ deformation
- Holographic dualities: the double trace version
- Holographic dualities: the single trace version
- Other solvable irrelevant deformations

(double-trace) $T\bar{T}$ deformations on the cylinder

[Zamolodchikov;Smirnov, Zamolodchikov; Cavaglia, Negro, Szecsenyi, Tateo; Cardy; Dubovsky, Flauger, Gorbenko; Dubovsky, Gorbenko, Mirbabayi; Conti, Iannella, Negro, Tateo; Frolov; …]

$$\frac{\partial S_\mu}{\partial \mu} = \int dx^2 \det T^\mu{}_\nu = \int dx^2 \,(T_{xx}T_{\bar{x}\bar{x}} - T_{x\bar{x}}T_{\bar{x}x})$$

$$x = \phi + t,\ \bar{x} = \phi - t,\ \phi \sim \phi + 2\pi R$$

$T_{\mu\nu}$: *stress tensor of the deformed theory at* μ

- solvable irrelevant deformation
- spectrum on a cylinder with $(x, \bar{x}) \sim (x + 2\pi R, \bar{x} + 2\pi R)$

$$E(\mu) = -\frac{R}{2\mu}\left(1 - \sqrt{1 + \frac{4\mu E}{R} + \frac{4\mu^2 J^2}{R^2}}\right), \quad J(\mu) = J$$

- modular invariance, connections to random metric, Nambu-Goto action, JT gravity, string theory, CDD factors…*[Review: Jiang]*

Free scalar CFT: $\mathscr{L}_0 = \partial\phi\bar{\partial}\phi$

stress tensor $T \propto \partial\phi\partial\phi, \quad \bar{T} \propto \bar{\partial}\phi\bar{\partial}\phi$

Infinitesimal $T\bar{T}$ deformation: $\delta\mathscr{L} = \mu T\bar{T}$

This deformation can be integrated and the full deformed Lagrangian is given by

$$\mathscr{L}_\mu = \frac{1}{2\mu}\left(\sqrt{4\mu\partial\phi\bar{\partial}\phi + 1} - 1\right) = -\frac{1}{2\mu} + \mathscr{L}_{NG}$$

$T\bar{T}$ deformed CFTs are shown to be modular invariant [Datta-Jiang]

$$Z\left(\frac{a\tau+b}{c\tau+d},\frac{a\bar{\tau}+b}{c\bar{\tau}+d};\frac{\hat{\mu}}{|c\tau+d|^2}\right)=Z(\tau,\bar{\tau};\hat{\mu}),\quad \hat{\mu}=\mu/R^2$$

Assuming large c and sparceness condition, the entropy in some parameter regime can be written as [Apolo-WS-Yu, WIP]

$$S=2\pi[\sqrt{\frac{c}{6}E_L(\mu)\left[1+2\mu E_R(\mu)\right]}+\sqrt{\frac{c}{6}E_R(\mu)\left[1+2\mu E_L(\mu)\right]}],\quad E_{L/R}=\frac{1}{2R}(E\pm J)$$

Alternative argument of the entropy formula: no level crossing in the spectrum

$$S_{T\bar{T}}(E_L(\mu),E_R(\mu))=S_{Cardy}(E_L,E_R),\quad E_{L/R}=\frac{1}{2R}(E\pm J)$$

$T\bar{T}$ with $\mu < 0$

The spectrum on a cylinder with $(x, \bar{x}) \sim (x + 2\pi R, \bar{x} + 2\pi R)$

$$E(\mu) = -\frac{R}{2\mu}\left(1 - \sqrt{1 + \frac{4\mu E}{R} + \frac{4\mu^2 J^2}{R^2}}\right), \quad J(\mu) = J, \qquad \mu < 0$$

- real energy for the ground state
- complex spectrum at very high energy
- high energy (but not too high) $S_{T\bar{T}}(E_L(\mu), E_R(\mu)) = S_{Cardy}(E_L, E_R), \quad E_{L/R} = \frac{1}{2R}(E \pm J)$

$$S = 2\pi[\sqrt{\frac{c}{6}E_L(\mu)\left[1 + 2\mu E_R(\mu)\right]} + \sqrt{\frac{c}{6}E_R(\mu)\left[1 + 2\mu E_L(\mu)\right]}]$$

c is the central charge before the deformation.

$T\bar{T}$ with $\mu > 0$

The spectrum on a cylinder with $(x, \bar{x}) \sim (x + 2\pi R, \bar{x} + 2\pi R)$

$$E(\mu) = -\frac{R}{2\mu}\left(1 - \sqrt{1 + \frac{4\mu E}{R} + \frac{4\mu^2 J^2}{R^2}}\right), \quad J(\mu) = J, \qquad \mu > 0$$

$6k$: central charge of the CFT

- ground state $E^{vac}(\mu) = -\frac{1}{2\mu}(1 - \sqrt{1 - \frac{c\mu}{3R^2}})$, complex if $\lambda \equiv \frac{c\mu}{6R^2} > \frac{1}{2}$

critical value: $\lambda_c = 1/2$

- high energy states always have real energies
- entropy $S = 2\pi[\sqrt{\frac{c}{6}E_L(\mu)\left[1 + \frac{2\mu}{R}E_R(\mu)\right]} + \sqrt{\frac{c}{6}E_R(\mu)\left[1 + \frac{2\mu}{R}E_L(\mu)\right]}]$
- Hagedorn growth at very high energy $E(\mu) \gg \frac{1}{\mu}$, $\quad S_{T\bar{T}} \sim 2\pi\sqrt{\frac{c\mu}{3}}E(\mu)$
- temperatures $T_{L/R} \equiv \left(\partial S_{T\bar{T}}/\partial E_{L/R}\right)^{-1}$, have an upper bound $T_L T_R \leq \frac{3}{4\pi^2 c\mu}$

[Apolo-Detournay-WS]

A single trace version of $T\bar{T}$ deformation

A single trace version of $T\bar{T}$ deformation can be defined for as a symmetric product $(\mathscr{M}_\mu)^p/S_p$, where the seed theory is a (double trace) deformed CFT_2.

- The spectrum in the twisted sector is given by

[Apolo-WS-Yu, WIP]

$$E_L^{(n)}(0) = E_L^{(n)}(\mu) + \frac{2\mu}{nR} R E_L^{(n)}(\mu) E_R^{(n)}(\mu)$$

- The entropy is

$$S^{single\ trace}\left(E_L, E_R\right) = 2\pi\left[\sqrt{\frac{c}{6} R E_L(\mu)\left[1 + \frac{2\mu}{Rp} E_R(\mu)\right]} + \sqrt{\frac{c}{6} R E_R(\mu)\left[1 + \frac{2\mu}{Rp} E_L(\mu)\right]}\right]$$

- The $T\bar{T}$ deformation
- Holographic dualities: the double trace version
- Holographic dualities: the single trace version
- Other solvable irrelevant deformations

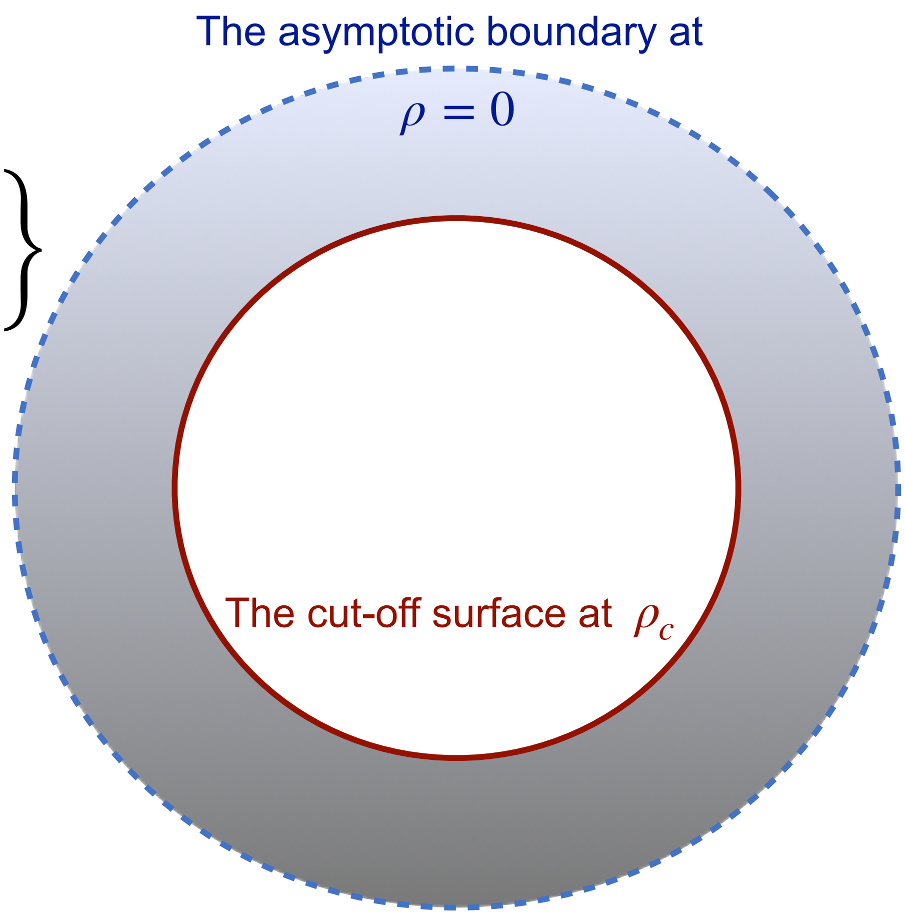
The holographic dualities for $T\bar{T}$ with $\mu < 0$

cutoff AdS_3 in Einstein gravity *[McGough-Mezei-Verlinde, Kraus-Liu-Marolf]*

Bananos metrics

$$ds^2 = \ell^2 \left\{ \frac{d\rho^2}{4\rho^2} + \frac{(du + \rho\overline{\mathscr{L}}(v)dv)(dv + \rho\mathscr{L}(u)du)}{\rho} \right\}$$

$$\varphi \sim \varphi + 2\pi,\ \ u = t + \varphi, v = \varphi - t,$$



- Einsstein gravity with Dirichlet boundary conditions

at the surface $g_{\varphi\varphi}(\rho, u, v) \equiv \ell^2 r_c^2 = -\dfrac{3}{c\mu}$,

$$ds^2 = \ell^2 \left\{ r_c^2(-dt'^2 + d\varphi'^2) + \mathscr{O}(r - r_c) \right\}$$

$$\boxed{\rho > \rho_c > 0}$$

- The $T\bar{T}$ deformed CFT on the cylinder $ds_c^2 = -dt'^2 + d\varphi'^2$

The holographic dualities for $T\bar{T}$ with $\mu < 0$

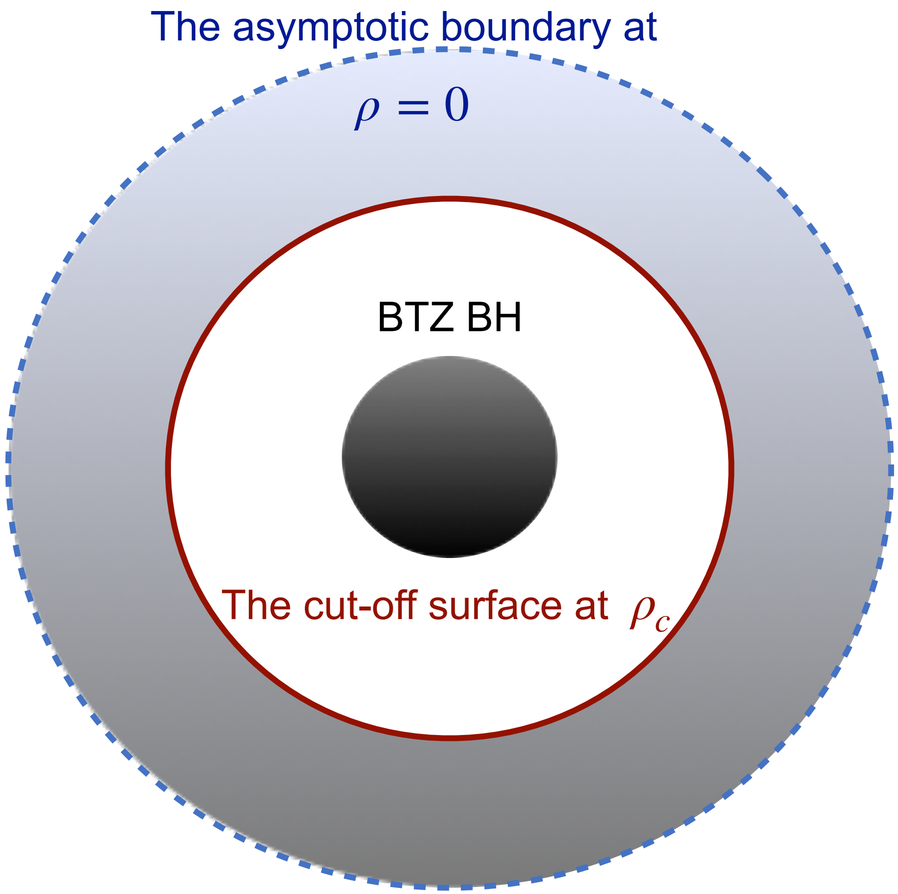
cutoff AdS_3 in Einstein gravity *[McGough-Mezei-Verlinde, Kraus-Liu-Marolf]*

evidence:

- quasi local energy of BTZ

= gravitational Noether charge of the time translation generator in the locally static frame $=Q(\partial_{t'})$

$\Leftrightarrow$deformed energy of $T\bar{T}$



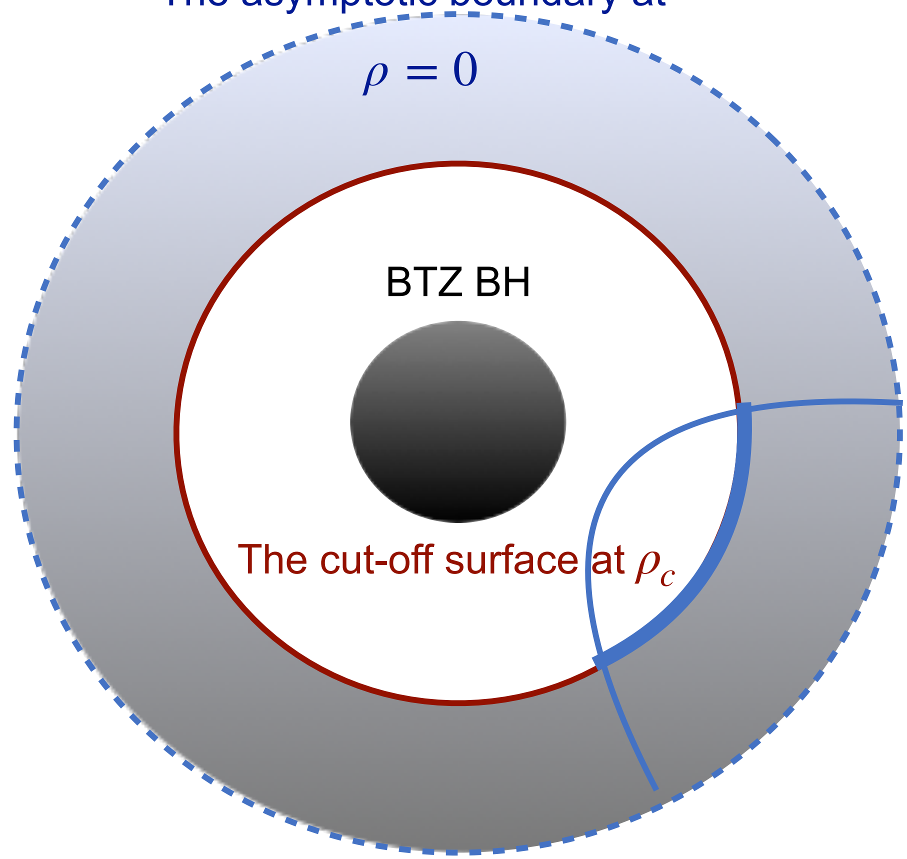
The holographic dualities for $T\bar{T}$ with $\mu < 0$

cutoff AdS_3 in Einstein gravity *[McGough-Mezei-Verlinde, Kraus-Liu-Marolf]*

- quasi local energy of BTZ
⇔deformed energy of $T\bar{T}$

- holographic entanglement entropy

[Donnelly-Shyam, Chen-Chen-Hao, Lewkowycz-Silverstein-Torroba]



Holographic dualities for $T\bar{T}$ with $\mu > 0$?

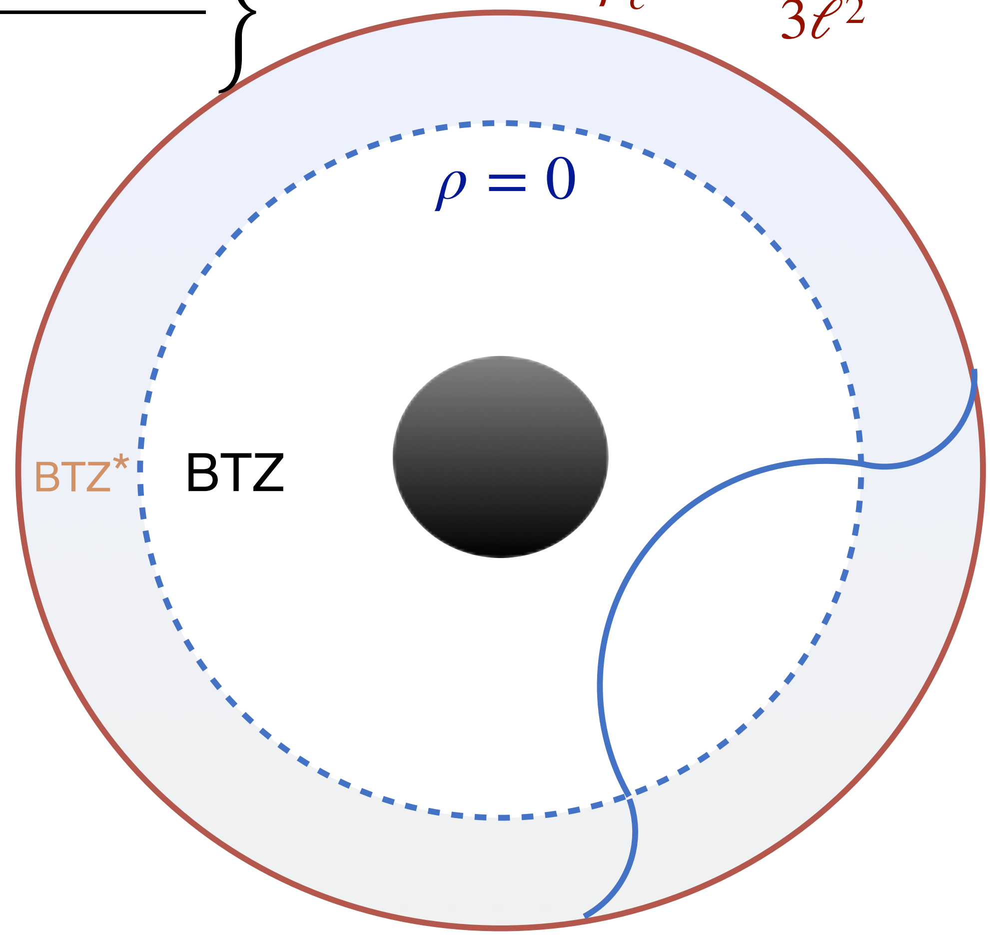
- Instead of cut-off, can we have a glue-on picture?

$$ds^2 = \ell^2 \left\{ \frac{d\rho^2}{4\rho^2} + \frac{(du + \rho\overline{\mathscr{L}}(v)dv)(dv + \rho\mathscr{L}(u)du)}{\rho} \right\}$$

$\rho > \rho_c(\mu, u, v)$

zero mass BTZ,

$\rho_c = -\frac{c\mu}{3\ell^2} < 0$



- quasi local energy of BTZ
$\leftrightarrow$deformed energy of $T\bar{T}$

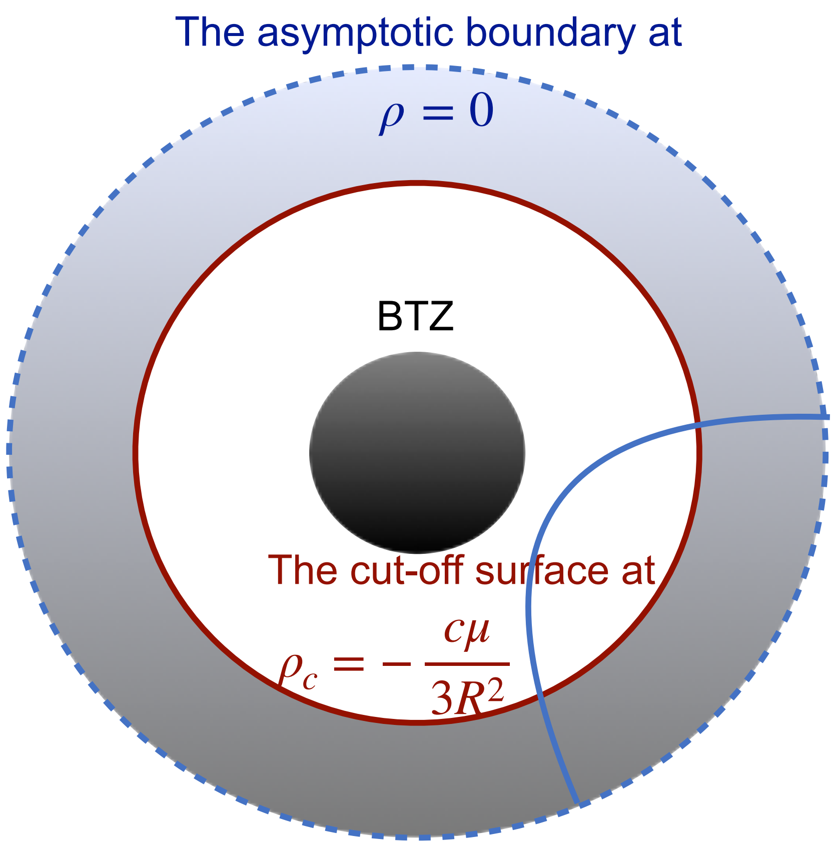
- holographic entanglement entropy

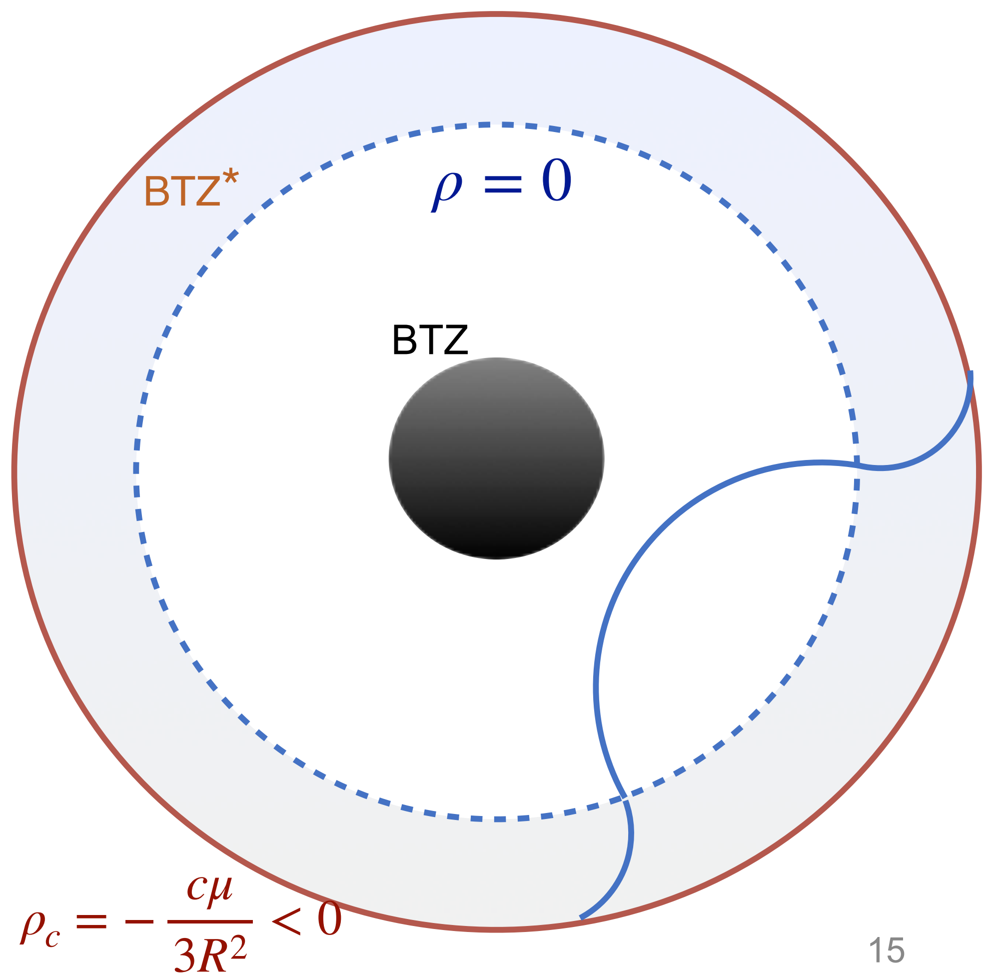
[Apolo-Hao-Lai-WS]

Holographic duality for (double-trace) $T\bar{T}$

- $\mu < 0 \leftrightarrow$ cut-off AdS_3
- $\mu > 0 \leftrightarrow$ glue-on AdS_3

- *'double trace'*
- *Universal*
- *local geometry unchanged*
- *changes the boundary condition*



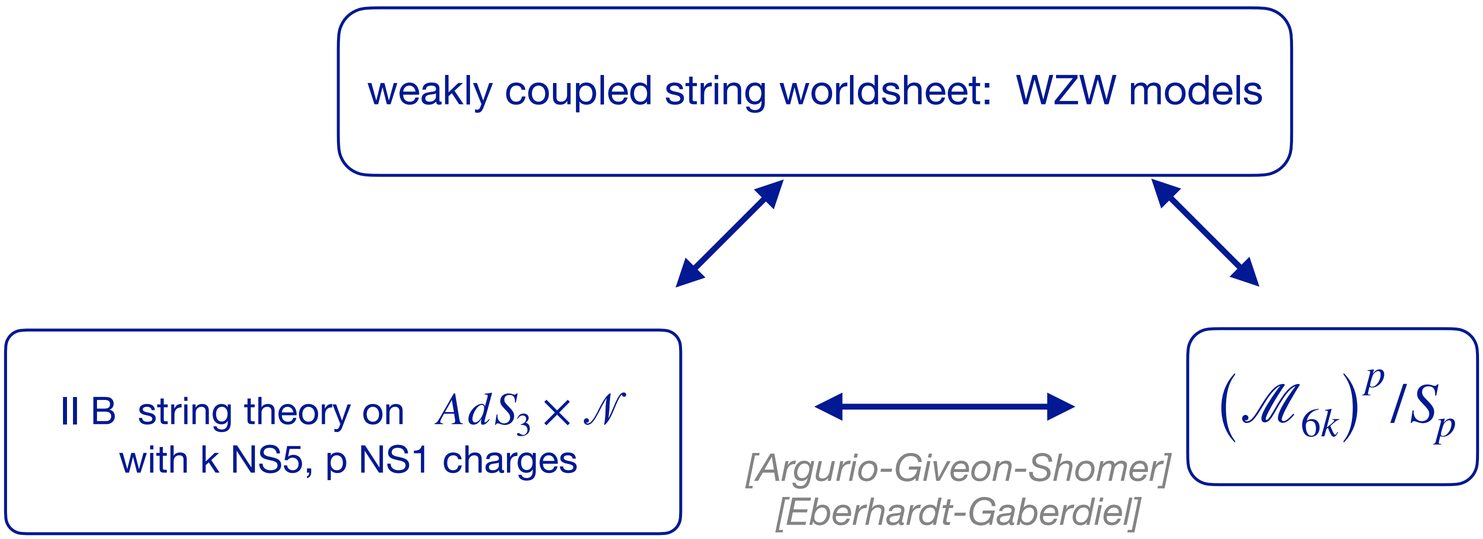


- The $T\bar{T}$ deformation
- Holographic dualities: the double trace version
- Holographic dualities: the single trace version
- Other solvable irrelevant

The holographic dual is no longer asymptotically locally AdS_3

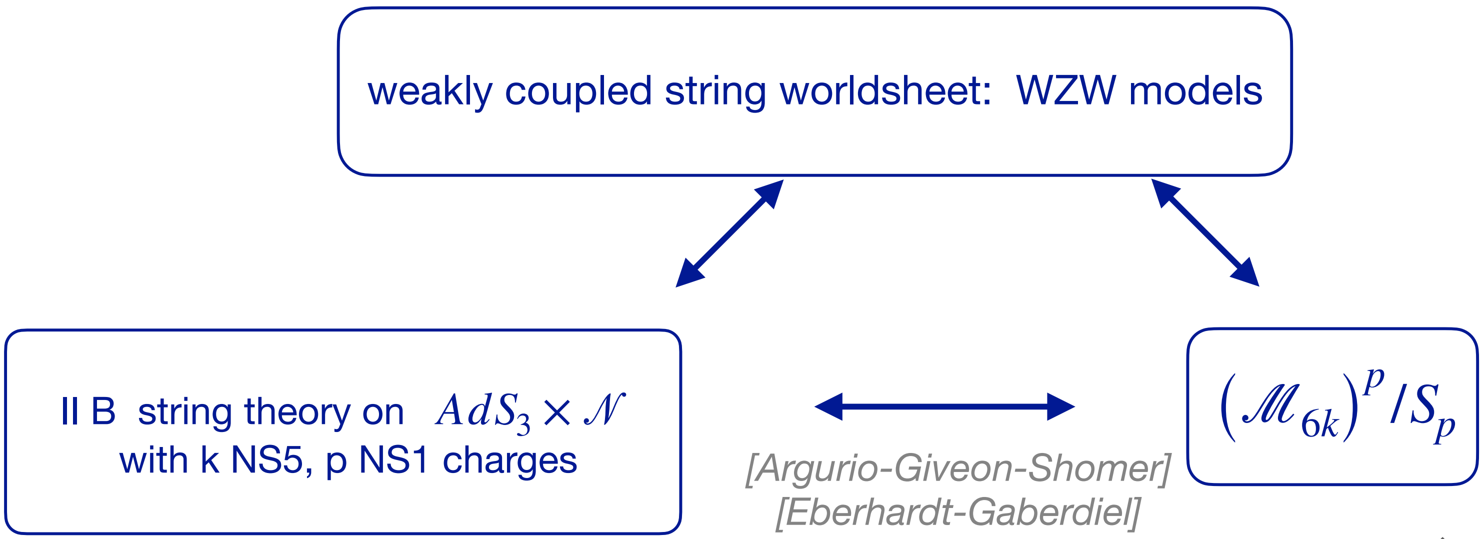
Holographic dualities for $T\bar{T}$: the single trace version

- starting from very specific example of AdS$_3$/CFT$_2$ in string theory



-

- starting from very specific example of AdS_3/CFT_2 in string theory



- a single trace version of $T\bar{T}$ deformation

$$\frac{\partial S^i_{\mathcal{M}_\mu}}{\partial \mu} = -4 \int J^i_{(1)} \wedge J^i_{(\bar{2})}$$

$\left(\mathcal{M}_\mu\right)^p / S_p$

TsT $\leftrightarrow T\bar{T}/J\bar{T}(T\bar{J})/J\bar{J}$

A conjecture *[Apolo-Detournay-WS]*

Starting from IIB string theory on locally $AdS_3 \times \mathcal{N}$ with NSNS background flux,

$$\mathrm{TsT}_{(X^m, X^{\bar{m}}; \hat{\mu})} \Longleftrightarrow \frac{\partial S_{\mathcal{M}_\mu}}{\partial \mu} = -4 \int J_{(m)} \wedge J_{(\bar{m})}$$

LHS: T-duality along X^m, then a shift $X^{\bar{n}} = X^{\bar{n}} - \hat{\mu}\, X^m$, and finally T-duality along X^m.

RHS: deformed symmetric product theory

Examples:

TsT with two $U(1)$s both in AdS_3 / one in AdS_3 and the other in $\mathcal{N}$ / both in $\mathcal{N}$

[Apolo-Detournay-WS] *[Chakraborty-Giveon-Kutasov; Apolo-WS]*

$\Updownarrow$

single trace $T\bar{T}$ / $J\bar{T}(T\bar{J})$ / $J\bar{J}$ deformations

[Giveon-Itzhaki-Kutasov, Giribet] *[Guica]*

TsT $\leftrightarrow T\bar{T}/J\bar{T}(T\bar{J})/J\bar{J}$

Side remark:

In IIB string theory on $AdS_5 \times S^5$ with RR background flux, *[Lunin-Maldacena]*

TsT with two $U(1)$s both in AdS_5 / one in AdS_5 and the other in S^5 / both in S^5

$\Updownarrow$

non-commutative / dipole / β deformations

Our conjecture

[Apolo, Detournay, WS]

Starting from IIB string theory on locally $AdS_3 \times \mathcal{N}$ with NSNS background flux,

TsT with two $U(1)$s both in AdS_3 / one in AdS_3 and the other in $\mathcal{N}$ / both in $\mathcal{N}$

$\Updownarrow$

single trace $T\bar{T}$ / $J\bar{T}(T\bar{J})$ / $J\bar{J}$ deformations

A conjecture

[Apolo-Detournay-WS]

Starting from IIB string theory on locally $AdS_3 \times \mathcal{N}$ with NSNS background flux,

$$\mathrm{TsT}_{\left(X^m, X^{\bar{m}}; \hat{\mu}\right)} \Longleftrightarrow \frac{\partial S_{\mathcal{M}_\mu}}{\partial \mu} = -4 \int J_{(m)} \wedge J_{(\bar{m})}$$

Evidence:

- action and symmetries
- string spectrum
- black hole thermodynamics

A useful rewriting of $T\bar{T}$ deformation

$$\frac{\partial S_\mu}{\partial \mu} = \int dx^2 \left(T_{xx}T_{\bar{x}\bar{x}} - T_{x\bar{x}}T_{\bar{x}x}\right)$$

$x = \phi + t,\ \bar{x} = \phi - t$

$T_{\mu\nu}$: *stress tensor of the deformed theory at* μ

$$\boxed{\frac{\partial S_\mu}{\partial \mu} = -4\int J_{(x)} \wedge J_{(\bar{x})}}$$

$J_{(x)}$: *Noether current that generates translation in* x

$J_{(\bar{x})}$: *Noether current that generates translation in* $\bar{x}$

$$T\bar{T}: \quad J_{(x)} = T_{xx}dx + T_{x\bar{x}}d\bar{x}, \quad J_{(\bar{x})} = T_{\bar{x}x}dx + T_{\bar{x}\bar{x}}d\bar{x}\,.$$

Solvable irrelevant deformations

- 1-parameter deformations:

$$\frac{\partial S_\mu}{\partial \mu} = -4 \int J_{(m)} \wedge J_{(\bar{m})}$$

$J_{(m)}/J_{(\bar{m})}$: Noether currents

chiral/anti-chiral at conformal point

$$T\bar{T}: \quad J_{(x)} = T_{xx} dx + T_{x\bar{x}} d\bar{x}, \quad J_{(\bar{x})} = T_{\bar{x}x} dx + T_{\bar{x}\bar{x}} d\bar{x}.$$

$$J\bar{T}: \quad J_{(n)} = J_x dx + J_{\bar{x}} d\bar{x}, \quad J_{(\bar{x})} = T_{\bar{x}x} dx + T_{\bar{x}\bar{x}} d\bar{x}.$$

$$T\bar{J}: \quad J_{(x)} = T_{xx} dx + T_{x\bar{x}} d\bar{x}, \quad J_{(\bar{n})} = J_x dx + J_{\bar{x}} d\bar{x}.$$

- 3-parameter deformations:

$$T\bar{T} + J\bar{T} + T\bar{J}: \quad \frac{\partial S_{\mu_0, \mu_+, \mu_-}}{\partial \mu_0} = -4 \int J_{(x)} \wedge J_{(\bar{x})}, \quad \frac{\partial S_{\mu_0, \mu_+, \mu_-}}{\partial \mu_+} = -4 \int J_{(n)} \wedge J_{(\bar{x})}$$

$$\frac{\partial S_{\mu_0, \mu_+, \mu_-}}{\partial \mu_-} = -4 \int J_{(x)} \wedge J_{(\bar{n})}$$

Solvable irrelevant deformations

- 1-parameter deformations:

$$\frac{\partial S_\mu}{\partial \mu} = -4\int J_{(m)} \wedge J_{(\bar{m})}$$

$J_{(m)}/J_{(\bar{m})}$: Noether currents

chiral/anti-chiral at conformal point

$$T\bar{T}: \quad J_{(x)} = T_{xx}dx + T_{x\bar{x}}d\bar{x}, \quad J_{(\bar{x})} = T_{\bar{x}x}dx + T_{\bar{x}\bar{x}}d\bar{x}.$$

$$J\bar{T}: \quad J_{(n)} = J_x dx + J_{\bar{x}}d\bar{x}, \quad J_{(\bar{x})} = T_{\bar{x}x}dx + T_{\bar{x}\bar{x}}d\bar{x}.$$

$$T\bar{J}: \quad J_{(x)} = T_{xx}dx + T_{x\bar{x}}d\bar{x}, \quad J_{(\bar{n})} = J_x dx + J_{\bar{x}}d\bar{x}.$$

- 3-parameter deformations:

$$T\bar{T} + J\bar{T} + T\bar{J}: \quad \frac{\partial S_{\mu_0,\mu_+,\mu_-}}{\partial \mu_0} = -4\int J_{(x)} \wedge J_{(\bar{x})}, \quad \frac{\partial S_{\mu_0,\mu_+,\mu_-}}{\partial \mu_+} = -4\int J_{(n)} \wedge J_{(\bar{x})}$$

$$\frac{\partial S_{\mu_0,\mu_+,\mu_-}}{\partial \mu_-} = -4\int J_{(x)} \wedge J_{(\bar{n})}$$

TsT $\leftrightarrow T\bar{T}/J\bar{T}(T\bar{J})/J\bar{J}$: the action

The string worldsheet action $S_{WS} = -\ell_s^{-2}\int d^2z\, M_{\mu\nu}\partial X^\mu\,\bar{\partial}X^\nu,$

$$M_{\mu\nu} \equiv G_{\mu\nu} + B_{\mu\nu}$$

$G_{\mu\nu}$: Target space metric
$B_{\mu\nu}$: NS-NS potential

TsT along $X^m, X^{\bar{m}}$: $M = \tilde{M}\left(I + 2\hat{\mu}\Gamma\tilde{M}\right)^{-1}$, $\Gamma_{\mu\nu} = \delta_\mu^m\delta_\nu^{\bar{m}} - \delta_\mu^{\bar{m}}\delta_\nu^m$ *$X^m, X^{\bar{m}}$ are isometries*

satisfies the differential equation: $\dfrac{\partial M}{\partial\hat{\mu}} = -2\ell_s^{-2}M\Gamma M$

TsT on string worldsheet can be formulated as : $\dfrac{\partial S_{WS}}{\partial\hat{\mu}} = -4\int \boldsymbol{j}_{(m)}\wedge \boldsymbol{j}_{(\bar{n})}$

$\boldsymbol{j}_{(m)}$, $\boldsymbol{j}_{(\bar{n})}$ *are* ***worldsheet*** *Noether 1-forms associated to ∂_{X^m}, and $\partial_{X^{\bar{m}}}$*

Noether charges $p_{(m)} \propto \oint \boldsymbol{j}_m$ *marginal deformation on the worldsheet*

---

irrelavent deformation on the dual theory

$T\bar{T}$ on the dual field theory $(\mathcal{M}_\mu)^p/S_p$: $\dfrac{\partial S_{\mathcal{M}_\mu}}{\partial\mu} = -4\int J_{(m)}\wedge J_{(\bar{m})}$

$J_{(m)}$, $J_{(\bar{m})}$ *are the* ***boundary spacetime*** *Noether 1-forms associated to ∂_{X^m}, and $\partial_{X^{\bar{m}}}$*

Noether charges $E_{(m)} \propto \sum_{i=1}^{p}\oint J_m^i$

Evidence for TsT $\leftrightarrow T\bar{T}/J\bar{T}(T\bar{J})/J\bar{J}$: the spectrum

After the TsT, string spectrum on a cylinder can be obtained by two observations:
[Apolo, WS;Apolo, WS; Apolo,Stephane,WS;]

1. String spectrum of AdS_3 on a cylinder with "winding"
$X^1(\sigma+2\pi)=X^1(\sigma+2\pi)+2\pi\gamma^1, X^{\bar{2}}(\sigma+2\pi)=X^{\bar{2}}(\sigma+2\pi)+2\pi\gamma^{\bar{2}}$ can be obtained from zero winding by "spectral flow" with parameter $\gamma^1/\gamma^{\bar{2}}$ in the left/right sector
[Maldacena,Ooguri]

2. TsT on the worldsheet $\Leftrightarrow$ field redefinition: *[Alday]*
string solutions on new background with periodic b.c.
$\Longleftrightarrow$ strings on the old background with twisted boundary conditions depending on the momentum $p_{(1)}, p_{(\bar{2})}$.
assuming $j_{(1)}/j_{(\bar{2})}$ to be chiral/antichiral up to total derivative terms (satisfied for the WZW model)
$\Longleftrightarrow$ momentum dependent "spectral flow" parameters

- The Virasoro constraints on AdS$_3$ background with winding w

$X^1(\sigma + 2\pi) = X^1(\sigma + 2\pi) + 2\pi w, X^{\bar{2}}(\sigma + 2\pi) = X^{\bar{2}}(\sigma + 2\pi) + 2\pi w$

is related to those of on AdS$_3$ background without winding

$$\hat{L}_0 = \tilde{L}_0 + wRp$$

Relation?

- The Virasoro constraints on the TsT background with winding

is related to those of on AdS$_3$ background by spectral flow transformations

$$\hat{L}_0 = \tilde{L}_0 + wRp(\hat{\mu}) + 2\hat{\mu}p(\hat{\mu})\bar{p}(\hat{\mu})$$

Relation between string spectra with winding w before and after the TsT transformations

$$= wRp(0)$$

$$\hat{L}_0 = \tilde{L}_0 + wRp(\hat{\mu}) + 2\hat{\mu}p(\hat{\mu})\bar{p}(\hat{\mu})$$

$$\hat{\bar{L}}_0 = \tilde{\bar{L}}_0 - wR\bar{p}(\hat{\mu}) - 2\hat{\mu}p(\hat{\mu})\bar{p}(\hat{\mu})$$

keep fixed

$$= -wR\bar{p}(0)$$

The dictionary $\mu = \ell^2\hat{\mu}$

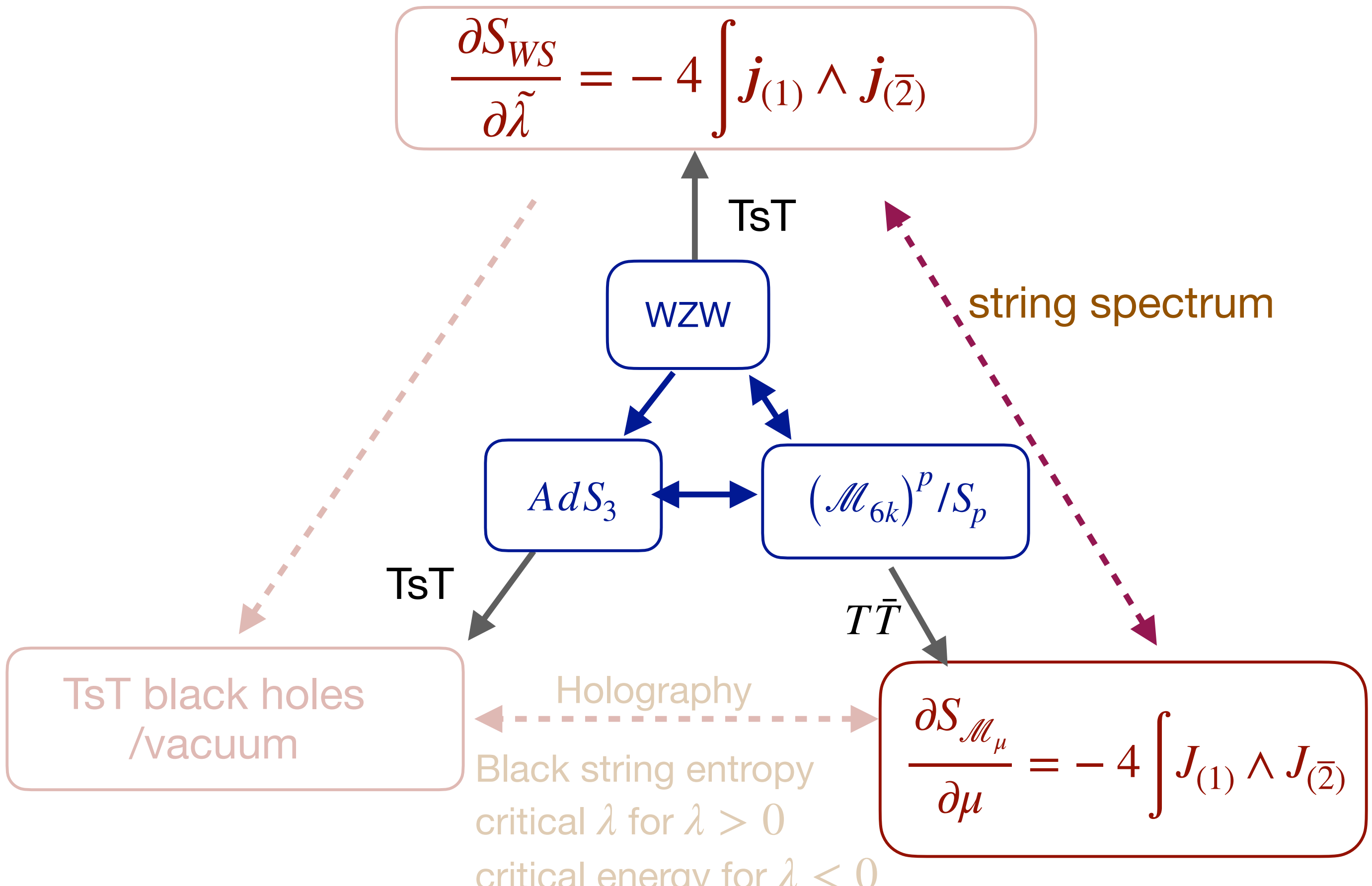
- $p \leftrightarrow \ell E_L, \quad \bar{p} \leftrightarrow -\ell E_R$
- w=-1: untwisted sector
- w<-1: twisted sector

Single trace $T\bar{T}$ deformation

$$E_L(0) = E_L(\mu) - \frac{2\mu}{w\ell}E_L(\mu)E_R(\mu),$$

$$E_R(0) = E_R(\mu) - \frac{2\mu}{w\ell}E_L(\mu)E_R(\mu)$$

TsT $\leftrightarrow T\bar{T}/J\bar{T}(T\bar{J})/J\bar{J}$



A two parameter family of classical solutions in IIB SUGRA with NSNS fluxes

$$d\tilde{s}_3^2 = \ell^2 \left\{ \frac{dr^2}{4\left(r^2 - 4T_u^2 T_v^2\right)} + r\,du\,dv + T_u^2 du^2 + T_v^2 dv^2 \right\}, \qquad (u, v) \sim (u + 2\pi, v + 2\pi)$$

$$e^{2\tilde{\Phi}} = \frac{k}{p}$$

→ *# of NS5 branes, magnetic charge*

→ *# of NS1 branes, electric charge*

- $T_u^2 \geq 0, T_v^2 \geq 0$: BTZ black holes
- $T_u = T_v = 0$: massless BTZ
- $T_u^2 < 0, T_v^2 < 0$: conical defect
- $T_u^2 = T_v^2 = -\frac{1}{4}$: global AdS_3

stationary solutions dual to the single trace $T\bar{T}$ deformed CFT_2 can be obtained from the solution via the following TsT transformations:

T-dualize along u, shifting $v \to v - \frac{2\lambda}{k} v$, and T-dualizing along u once more

$$\frac{ds_3^2}{\ell^2} = \frac{dr^2}{4\left(r^2 - 4T_u^2 T_v^2\right)} + \frac{r\,du\,dv + T_u^2 du^2 + T_v^2 dv^2}{1 + 2\lambda r + 4\lambda^2 T_u^2 T_v^2}$$

$$e^{2\Phi} = \frac{k}{p} \left(\frac{1 - 4\lambda^2 T_u^2 T_v^2}{1 + 2\lambda r + 4\lambda^2 T_u^2 T_v^2} \right) e^{-2\phi_0} \qquad (u, v) \sim (u + 2\pi, v + 2\pi)$$

In string frame, the TsT transformed solution is

$$\frac{ds_3^2}{\ell^2} = \frac{dr^2}{4\left(r^2 - 4T_u^2 T_v^2\right)} + \frac{r du dv + T_u^2 du^2 + T_v^2 dv^2}{1 + 2\lambda r + 4\lambda^2 T_u^2 T_v^2} \qquad (u, v) \sim (u + 2\pi, v + 2\pi)$$

$$e^{2\Phi} = \frac{k}{p}\left(\frac{1 - 4\lambda^2 T_u^2 T_v^2}{1 + 2\lambda r + 4\lambda^2 T_u^2 T_v^2}\right) e^{-2\phi_0}$$

- λ is the deformation parameter
- T_u, T_v parameterize the phase space of the theory
- The resulting geometry interpolates

IR: locally AdS

UV$(r \to \infty)$: $ds^2 \propto dy^2 + du dv,\ \partial_y \Phi \propto \lambda, y \propto \ln r$

This is asymptotically flat spacetime in the string frame!

- $T_u = T_v = 0, \phi_0 = 0,$ the LST background of *[Giveon, Itzhaki, Kutasov]*
- $T_u = T_v = 0, \phi_0 = 0, \lambda = 1/2,$ black string *[Horne-Horowitz]*

The holographic dictionary before deformation

$$d\tilde{s}_3^2 = \ell^2 \left\{ \frac{dr^2}{4\left(r^2 - 4T_u^2 T_v^2\right)} + r du dv + T_u^2 du^2 + T_v^2 dv^2 \right\}, \qquad (u, v) \sim (u + 2\pi, v + 2\pi)$$

$$e^{2\tilde{\Phi}} = \frac{k}{p}$$

- Brown-Henneaux central charge $c = 6pk$
- Noether charges $\mathcal{Q}_{\partial_u} \leftrightarrow E_L$

- coordinates $u \leftrightarrow x, \quad v \leftrightarrow \bar{x};$
- gravitational Noether charge of $\partial_u \leftrightarrow$ Noether charge of ∂_x

$$\mathcal{Q}_{\partial_u} = \frac{1}{2}(M + J) = \frac{c}{6}T_u^2 \leftrightarrow E_L$$

- global AdS_3, *at* $T_u = T_v = \frac{i}{2}$ $\leftrightarrow$ NS vaccum on the cylinder $E_L = E_R = -\frac{c}{24}$
- zero mass BTZ($T_u = T_v = 0$) $\leftrightarrow$ Ramond vacuum
- Finite temperature $\qquad \ell T_L = \frac{T_u}{\pi}, \quad \ell T_R = \frac{T_v}{\pi}$
- The Bekenstein-Hawking entropy $\leftrightarrow$ Cardy formula in the dual CFT_2

The ground state can be obtained by assuming $T_u^2 = T_v^2 \equiv -\rho_0^2, \quad \rho_0 > 0$ and requiring smoothness at the origin. After a change of coordinate, the metric is

$$ds_3^2 = \ell^2 \left\{ \frac{d\rho^2}{\rho^2+1} + \frac{\rho^2 d\varphi^2 - \left(\rho^2+1\right) dt^2}{1+2\lambda\rho^2} \right\}$$

$$\rho \in [0,\infty), \quad \varphi \sim \varphi + 2\pi$$

$$\rho_0 = \frac{1}{2\lambda}(1-\sqrt{1-2\lambda})$$

$$B = -\frac{\ell^2\left(\rho^2+\rho_o\right)}{2(1+2\lambda\rho^2)} du \wedge dv, \quad e^{2\Phi} = \frac{k}{p}\frac{\left(1-2\lambda\rho_o^2\right)}{2\rho_o\left(1+2\lambda\rho^2\right)} e^{-2\phi_0}$$

- $\lambda > \frac{1}{2}$ complex solution;
- $\lambda_c = \frac{1}{2}, \ e^{2\Phi} \to 0$ unless ϕ_0 is fine tuned, infinitely weak string coupling everywhere
- $0 < \lambda < \frac{1}{2}$, smooth and real solution

  IR: $\rho \to 0$, global AdS_3 , smooth, no horizons
  UV: $R^{1,1} \times S^1$, locally flat with linear dilaton, infinitely weak coupled strings

- $\lambda < 0$, CTC and curvature singularity at $\rho_c^2 = 1/2\,|\lambda\,|$

The ground state can be obtained by assuming $T_u^2 = T_v^2 \equiv -\rho_0^2, \quad \rho_0 > 0$ and requiring smoothness at the origin. After a change of coordinate, the metric is

$$ds_3^2 = \ell^2 \left\{ \frac{d\rho^2}{\rho^2+1} + \frac{\rho^2 d\varphi^2 - \left(\rho^2+1\right) dt^2}{1+2\lambda\rho^2} \right\}$$

$$\rho \in [0,\infty), \quad \varphi \sim \varphi + 2\pi$$

$$\rho_0 = \frac{1}{2\lambda}(1-\sqrt{1-2\lambda})$$

$$B = -\frac{\ell^2\left(\rho^2+\rho_o\right)}{2(1+2\lambda\rho^2)} du \wedge dv, \quad e^{2\Phi} = \frac{k}{p} \frac{\left(1-2\lambda\rho_o^2\right)}{2\rho_o\left(1+2\lambda\rho^2\right)} e^{-2\phi_0}$$

TsT/$T\bar{T}$ matching: $\ell \leftrightarrow R, \quad \lambda\ell_s^2 \leftrightarrow \mu$
Noether charges $\mathcal{Q}_{\partial_u} \leftrightarrow E_L^{vac}$

- $\lambda > \frac{1}{2}$ complex solution;
- $\lambda_c = \frac{1}{2}$, $e^{2\Phi} \to 0$ unless ϕ_0 is fine tuned, infinitely weak string coupling everywhere
- $0 < \lambda < \frac{1}{2}$, smooth and real solution
  IR: $\rho \to 0$, global AdS_3 , smooth, no horizons
  UV: $R^{1,1} \times S^1$, locally flat with linear dilaton, infinitely weak coupled strings

- $\lambda < 0$, CTC and curvature singularity at $\rho_c^2 = 1/2|\lambda|$

Black holes $T_u, T_v > 0$, asymptotic to $R^{1,1} \times S^1$ at large radius.
Horizon at $r_+ = 2T_uT_v$, independent of λ
Electric and magnetic charges $Q_e = pe^{2\phi_0}, \quad Q_m = k$

TsT/$T\bar{T}$ dictionary for $\phi_0 = 0$, i.e. fixed Q_e, Q_m

- Noether charges $\mathcal{Q}_{\partial_u} \leftrightarrow E_L$
- smooth Euclidean geometry at the horizon ↔ torus parameters

$$(u, v) \sim \left(u + i/\ell T_L, v - i/\ell T_R\right) \qquad \ell T_L = \frac{1}{\pi}\frac{T_u}{1 + 2\lambda T_u T_v}, \quad \ell T_R = \frac{1}{\pi}\frac{T_v}{1 + 2\lambda T_u T_v}$$

- Bekenstein Hawking entropy ↔entropy of single trace $T\bar{T}$

$$S_{TsT} = 2\pi\{\sqrt{\mathcal{Q}_{\partial_u}\left(Q_eQ_m + 2\lambda\mathcal{Q}_{-\partial_v}\right)} + \sqrt{\mathcal{Q}_{-\partial_u}\left(Q_eQ_m + 2\lambda\mathcal{Q}_{\partial_u}\right)}\} = S_{T\bar{T}}$$

- $T_u = T_v = \dfrac{i}{2\lambda}(1 - \sqrt{1 - 2\lambda}) \leftrightarrow$ ground state, NS vacuum
- $T_u = T_v = 0 \leftrightarrow$ Ramond vacuum
- upper bound for the temperatures $T_LT_R \leq \dfrac{1}{8\pi\ell^2\lambda}$ by requiring real dilaton

↔ upper bound for the temperatures in $T\bar{T}$ deformation

How about $\lambda < 0$?

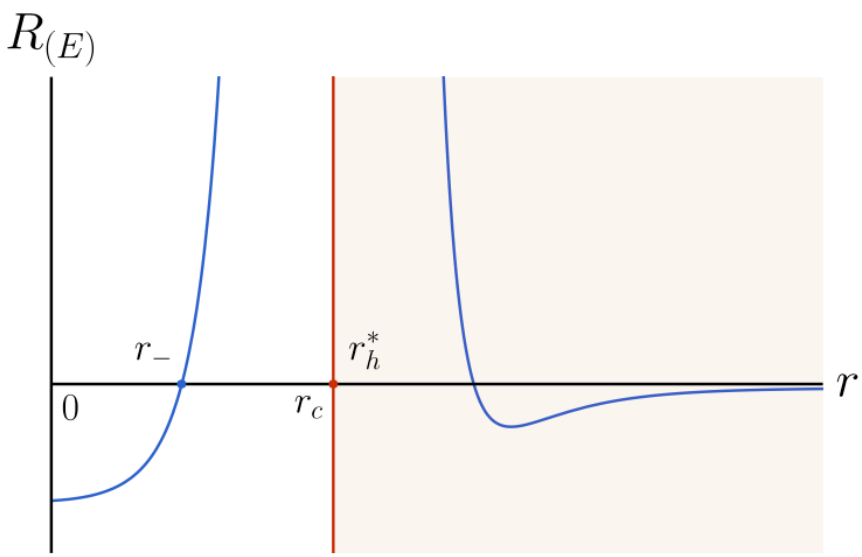
- CTC & conical singularity at large radius $r_c = \frac{1}{2\lambda} + 2\lambda T_u^2 T_v^2$

- real dilaton $\leftrightarrow$ real spectrum

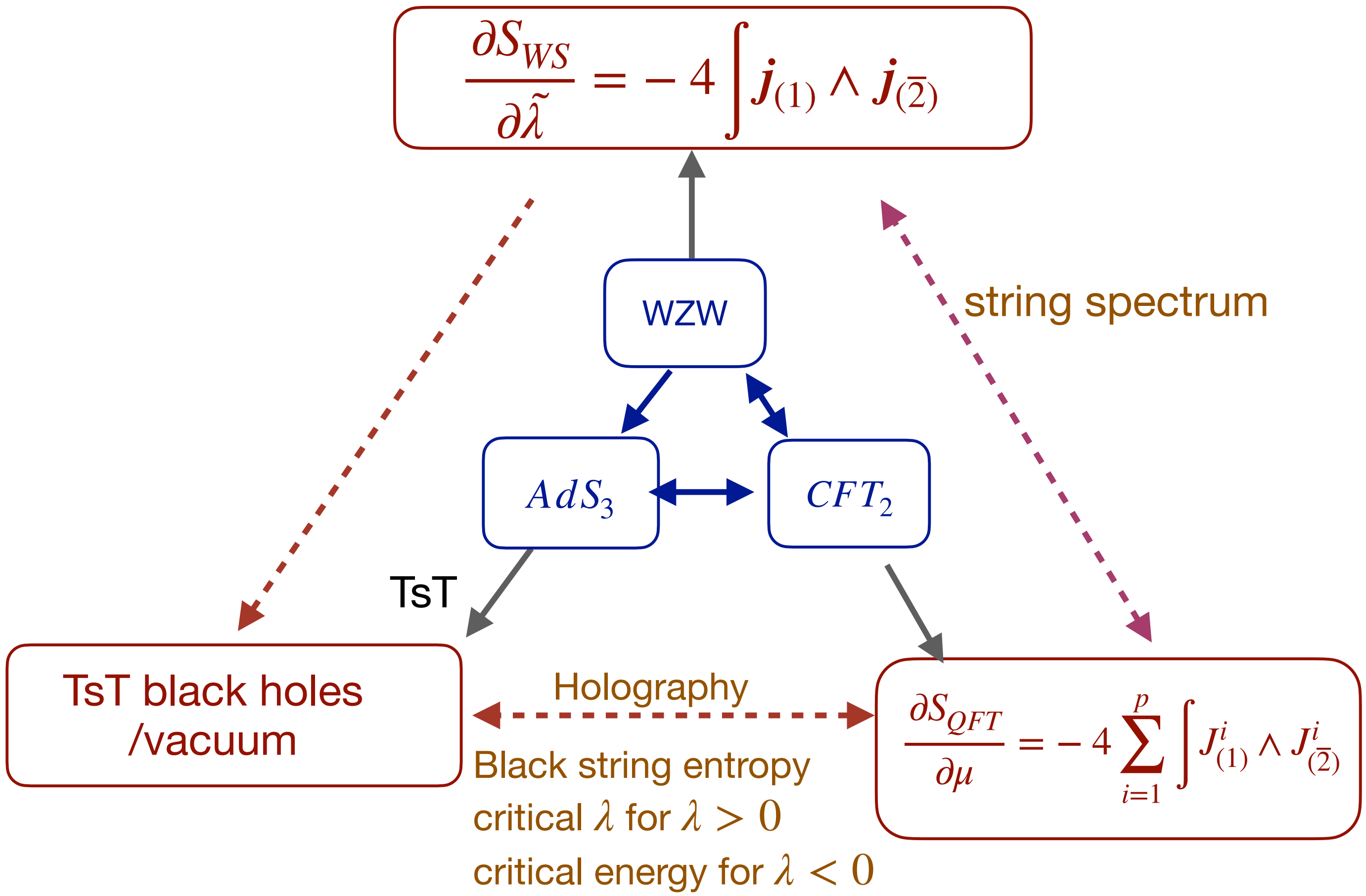
$$T_u T_v \leq \frac{1}{2|\lambda|} \leftrightarrow E(\mu) \leq \frac{pR^2}{2|\mu|}$$

"=" when the horizon coincides with the CTC & conical singularity

- Region of positive cosmological constant



A class of toy models for non-AdS holography



- The $T\bar{T}$ deformation
- Holographic dualities: the double trace version
- Holographic dualities: the single trace version
- Other solvable irrelevant deformations

Another example

An explicit and tractable toy model for Kerr/CFT in string theory

*[Azeyanagi-Hofman-WS-Strominger 13'
Apolo-WS 18', 19', Chakraborty-Giveon-Kutasov 18']*

spacetime: $AdS_3 \times S^3 \times M^4$ (w/ NSNS flux) —TsT along→ $WAdS_3 \times_w S^3 \times M^4$

⇕ ⇕

worldsheet: WZW —marginal deformation→ $\partial_\lambda S_{WS} \propto \int d\sigma^2 j^- \bar{k}^3$

⇕ ⇕

holographic dual: $(M_{6k})^p / S_p$ —single trace $J\bar{T}$ deformation→ $(M_\mu)^p / S_p, \quad \partial_\mu S^i \propto \int dx^2 J_i \bar{T}_i$

More general examples

[Chakraborty-Giveon-Kutasov 19', Apolo-WS, 21']

spacetime: $AdS_3 \times S^3 \times M^4$ (w/ NSNS flux) → three TsT transformations → new black hole solutions

⇕ ⇕

worldsheet: WZW → marginal deformation →

$$\partial_{\lambda_0} S_{WS} = -4 \int j_{(1)} \wedge j_{(\bar{2})}$$

$$\partial_{\lambda_+} S_{WS} = -4 \int j_{(3)} \wedge j_{(\bar{2})} \quad \partial_{\lambda_+} S_{WS} = -4 \int j_{(1)} \wedge j_{(\bar{4})}$$

⇕ ⇕

holographic dual: $(M_{6k})^p / S_p$ → single trace $T\bar{T} + J\bar{T} + T\bar{J}$ deformations → $(M_{\lambda_0, \lambda_+, \lambda_-})^p / S_p$

Holographic dualities for irrelevant deformations

Holography for ``double trace” deformations

- *‘double trace’*
- *Universal*
- *local geometry unchanged*
- *changes the boundary condition*

- $d = 2,\ T\bar{T}(with\ \mu < 0) \leftrightarrow$ cut-off AdS_3 in Einstein gravity *[McGough-Mezei-Verlinde]*
- $d = 2,\ T\bar{T}(with\ \mu > 0) \leftrightarrow$ glue-on AdS_3 in Einstein gravity *[Apolo-Hao-Lai-WS, WIP]*
- $d = 2,\ \ T\bar{T} + \Lambda_2 \leftrightarrow$ patch of dS *[Gorbenko-Silverstein-Torroba]*
- $d > 2,\ T\bar{T} \leftrightarrow$ cutoff AdS_{d+1} in Einstein gravity *[Hartman-Kruthoff-Shaghoulian-Tajdini,Taylor]*
- $d = 1,\ T\bar{T} \leftrightarrow$ cutoff JT gravity *[Gross-Kruthoff-Rolph-Shaghoulian, Iliesiu-Kruthoff-Turiaci-Verlinde, Stanford-Yang]*
- $d = 2,\ \ J\bar{T} \leftrightarrow$ AdS_3 in Einstein gravity+Chern-Simons gauge theory *[Bzowski-Guica]*

Holography for ``single trace” deformations

- *‘single trace’*
- *embedded in string theory*
- *asymptotic geometry changed*

- $d = 2,\ \ T\bar{T} \leftrightarrow$ *black strings* in string theory *[Giveon-Itzhaki-Kutasov, Apolo-Detournay-WS]*
- $d = 2,\ \ J\bar{T} \leftrightarrow$ *WAdS*$_3$ in string theory [Chakraborty-Giveon-Kutasov; Apolo-WS]
- $d = 2,\ \ T\bar{T} + J\bar{T} + T\bar{J} \leftrightarrow$ *three TsT transformations in string theory* [Chakraborty-Giveon-Kutasov; Apolo-WS]

Thank you!